Resonant excitation of trapped coastal waves by free inertia-gravity waves and its effects on transport and mixing

V. Zeitlin

[1]Laboratory of Dynamical Meteorology, Paris, France

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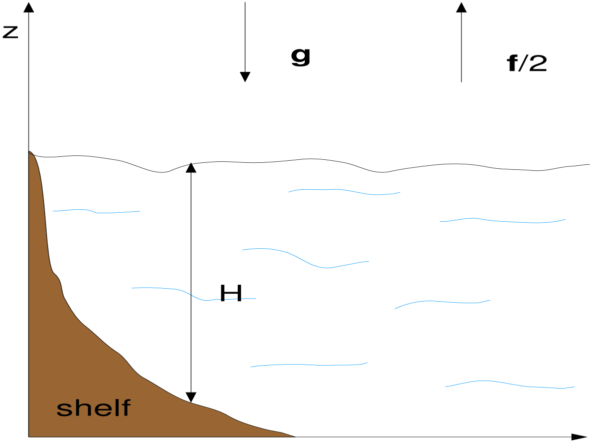
Ideology

- Trapped waves in the coastal waveguide interact with free inertia-gravity waves
- Cubic nonlinearity $\rightarrow$ interaction triadic
- Interaction may be resonant $\leftrightarrow$ resonant triads
- If so, resonant excitation of trapped waves by free waves, with two possible mechanisms:
  1. Free wave $\rightarrow$ Trapped wave + Trapped wave ($F \rightarrow TT$)
  2. Free wave + Free wave $\rightarrow$ Trapped wave ($F + F \rightarrow T$)
- Resonant growth of the trapped wave + nonlinearity/dissipation $\Rightarrow$ coherent/dissipative structures

Remark

Twist: infinitely long trapped wave $\equiv$ Coastal current
$\Rightarrow F + C \rightarrow T$ resonance.

Setup: one- or two-layer rotating shallow water model



Steep vs gentle bathymetry

Steep slopes

Characteristic horizontal scale of bathymetry $<< R_d \Rightarrow$ Trapped waves (leading order): (almost) non-dispersive Kelvin waves. Possible resonances:
$F + F \to T$, $F + C \to T(baroclinic\, case)$.

Gentle slope

Characteristic horizontal scale of bathymetry $\mathcal{O}\left(R_d\right) \Rightarrow$ Trapped waves: dispersive shelf and edge waves. Details of bathymetry unimportant: spectrum of coastal waves universal, Huthnance, 1975. All resonances possible.

Non-dimensional RSW equations with idealized coast

$$\begin{aligned} u_t - v + h_x &= -\epsilon(uu_x + vu_y) \\ v_t + u + h_y &= -\epsilon(uv_x + vv_y) \\ h_t + u_x + v_y &= -\epsilon\left((hu)_x + (hv)_y\right). \end{aligned}$$

Boundary condition: $x \geq 0, \quad u|_{x=0} = 0.$

Scaling

Time: f^{-1}; Space: $L \sim R_d = \frac{\sqrt{gH}}{f}$; Velocity: U.
Rossby number $Ro \equiv \epsilon = \frac{U}{fL}$.

Linear waves $(u, v, h) \propto e^{i(\sigma t - kx - ly)}$:

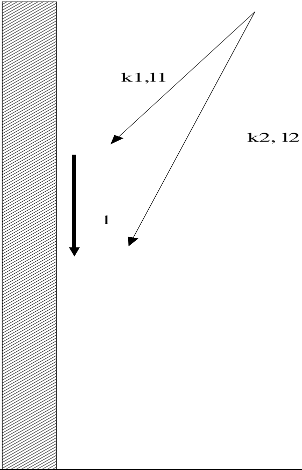
- Free inertia-gravity waves (IGW) with dispersion

$$\sigma = \sqrt{1 + k^2 + l^2}$$

- Trapped non-dispersive Kelvin waves (KW) with

$$\sigma = -l, \quad k = -i$$

IGW - KW interaction



Conditions of IGW - KW resonance

A pair of IGW with frequencies $\sigma_{1,2}$ and along-coast wavenumbers $l_{1,2}$ is in resonance with a KW with wavenumber l if ("difference" resonance)

$$\sigma_1 - \sigma_2 = \sigma_K = -l, \quad l_1 - l_2 = l, \; l \neq 0. \tag{1}$$

For $l < 0$:

$$|l| = \sqrt{1 + k_1^2 + l_1^2} - \sqrt{1 + k_2^2 + l_2^2}, \;\; l_2 = l_1 + |l|\,, \tag{2}$$

and

$$\sqrt{1 + k_1^2 + l_1^2} - |l| = \sqrt{1 + k_2^2 + (l_1 + |l|)^2}. \tag{3}$$

Evolution equation for KW

Killing resonances in RSW equations with coast with the help of slow-time corrections $\Rightarrow$

$$K_T + KK_\eta = Se^{il\eta} + S^* e^{-il\eta}, \tag{4}$$

where $\eta = y + t$,

$$\begin{aligned} S &= \int_0^\infty dx\, e^{-x} \left[(H_1 U_2^* + U_1 H_2^*)_x - U_1 V_{2_x}^* - V_{1_x} U_2 \right. \\ &+ \left. il(H_1 V_2^* + V_1 H_2^* - V_1 V_2^*) \right], \end{aligned} \tag{5}$$

and (U_i, V_i, H_i), $i = 1, 2$ are amplitudes of 2 free waves.

Final form of the evolution equation

From the polarisation relations get:

$$S = iA_1 A_2 s, \quad Im(s) = 0 \tag{6}$$

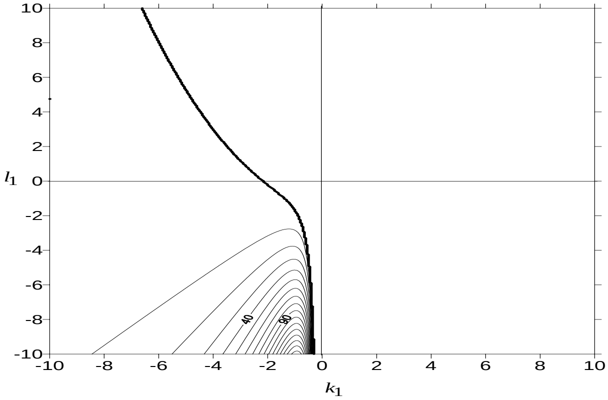
and hence

$$K_T + KK_\eta = -2sA_1 A_2 \sin l\eta, \tag{7}$$

- harmonically forced Hopf equation

$$\begin{aligned} s \;=\; & \frac{4l}{(k_1^2+1)(k_2^2+1)[1+(k_1+k_2)^2][1+(k_1-k_2)^2]} \times \\ & \Big[(\sigma_1 l_2 + \sigma_2 l_1 - l_1 l_2)(1+k_1^2+k_2^2) \\ & + \sigma_2 l_1 k_1 (1+k_1^2-k_2^2) + \sigma_1 l_2 k_2 (1+k_2^2-k_1^2) \\ & + 2k_1 k_2 (l_1 l_2 - (1+k_1^2)(1+k_2^2))\Big] \end{aligned} \tag{8}$$

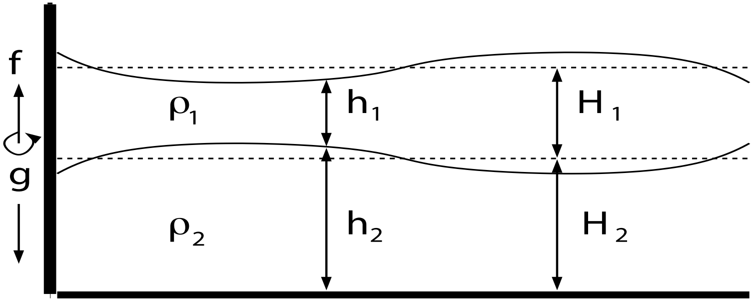
Isopleths of the interaction coefficient $s(l, k_1, l_1)$ for $l = -1$ at the interval 10



Comments

- Equation (7) is completely integrable in Lagrangian coordinates. Main property of solutions: breaking and formation of Kelvin fronts $\rightarrow$ a route to dissipation in the ocean
- The mechanism produces coastal wave "from nothing"
- Small deviations from infinite slope - weak dispersion $\Rightarrow$ forced Hopf equation $\rightarrow$ forced KdV equation, also integrable
- Details: Reznik & Zeitlin, Phys. Letters A, 2009.

2-layer RSW with idealized coast



Non-dimensional equatins of motion

$$
\begin{aligned}
\left(\partial_t + \epsilon \vec{v}_1 \cdot \vec{\nabla}\right) \vec{v}_1 + f\hat{z} \wedge \vec{v}_1 + \vec{\nabla}\left(\eta_1 + \eta_2\right) &= 0, \\
\partial_t \eta_1 + \vec{\nabla} \cdot \left(\left(d + \epsilon\eta_1\right)\vec{v}_1\right) &= 0, \\
\left(\partial_t + \epsilon \vec{v}_1 \cdot \vec{\nabla}\right) \vec{v}_2 + f\hat{z} \wedge \vec{v}_2 + \vec{\nabla}\left(r\eta_1 + \eta_2\right) &= 0, \\
\partial_t \eta_2 + \vec{\nabla} \cdot \left(\left(1 - d + \epsilon\eta_2\right)\vec{v}_2\right) &= 0.
\end{aligned}
\tag{9}
$$

Scaling: same as in 1-layer RSW. New parameters: $d = \frac{H_1}{H_1+H_2}$, $r = \frac{\rho_1}{\rho_2} \leq 1$.

Baroclinic-barotropic decomposition and wave spectrum

$$d = \frac{1}{2} \to \vec{v}^{\pm} = \sqrt{r}\vec{v}_1 \pm \vec{v}_2, \quad \eta^{\pm} = 2\left(\sqrt{r}\eta_1 \pm \eta_2\right) \Rightarrow$$

Wave spectrum

- Free IGW with dispersion relation:

$$\sigma^2_{IG\pm} = 1 + c^2_{\pm}\vec{k}^2, \ \vec{k} = (k, l), \tag{10}$$

- Trapped dispersionless Kelvin (K) waves propagating along the y axis with the phase velocity $c_{\pm}$:

$$\sigma^2_{K\pm} = c_{\pm} l^2. \tag{11}$$

$c_{\pm} = \sqrt{\frac{1 \pm \sqrt{r}}{2}}$ - dimensionless phase velocities of BT and BC GW

Wave and mean-current solutions

Waves

$$(u^{\pm}_{IG}, v^{\pm}_{IG}, \eta^{\pm}_{IG}) = \left(U^{\pm}(x), V^{\pm}(x), N^{\pm}(x)\right) e^{i(ly-\sigma t)} + c.c., \quad (12)$$

$$(u^{\pm}_{K}, v^{\pm}_{K}, \eta^{\pm}_{K}) = (0, c_{\pm}K^{\pm}(y + c_{\pm}t), -K^{\pm}(y + c_{\pm}t))e^{-x/c_{\pm}}, \quad (13)$$

$K^{\pm}$ - arbitrary functions of arguments $\xi^{\pm} = y + c_{\pm}t$

Mean current

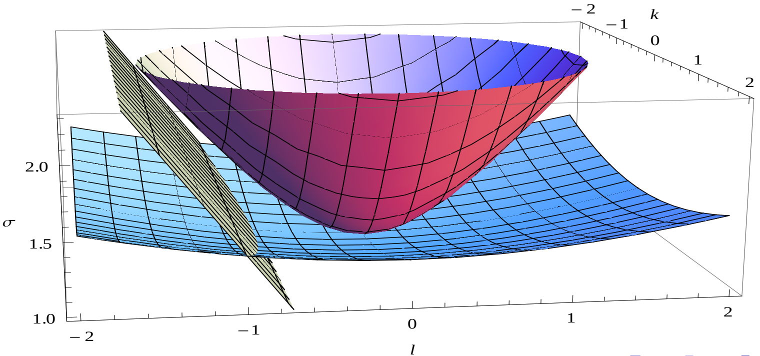
$$u^{\pm}_{M} = 0, \quad v^{\pm}_{M} = 1/2(1 \pm \sqrt{r})\partial_x \eta^{\pm}_{M}. \quad (14)$$

Coastal current: $v_{1,2}$ rapidly decaying. Surface current:

$$v_{2_M} = 0, \Rightarrow r\eta_{1_M} + \eta_{2_M} = 0. \quad (15)$$

Dispersion relation for 2-layer RSW in a half-plane

Wave spectrum: barotropic (BT) (faster) and baroclinic (BC) (slower) versions of each kind of waves: IGW and KW (baroclinic KW not shown).



What is new due to baroclinicity?

- IGW+IGW $\rightarrow$ KW resonances:

$$l_{IG_1} \pm l_{IG_2} = l_K, \quad \sigma_{IG}(\vec{k}_1) \pm \sigma_{IG}(\vec{k}_2) = \sigma_K(l),$$

  1. BT KW may be excited by BT - BT, BT + BC, and BC + BC IGW ("difference" and "sum" resonances)
  2. BC KW may be excited by BT - BT, BT - BC, and BC - BC IGW (only "difference" resonances)
- New resonance: BC IGW + Coastal Current (CC) $\rightarrow$ BT KW due to the intersection of dispersion surfaces

Remark

BC + BC IGW and BC IGW + CC excitation: weak wave signature at the surface $\Rightarrow$ strong surface signature of the excited wave ("wave from nothing").

Analysis of resonance conditions: wave-wave

"Difference" resonance conditions: similar to 1-layer case

"Sum" resonance conditions:

$$l_{IG_1} + l_{IG_2} = l_K, \quad \sigma_{IG}(\vec{k}_1) + \sigma_{IG}(\vec{k}_2) = \sigma_K(l) \Rightarrow \tag{16}$$

$$l_1 + l_2 = l = -|l|, \quad \sqrt{1 + c_-^2(k_1^2 + l_1^2)} + \sqrt{1 + c_-^2(k_2^2 + l_2^2)} = c_+|l|, \tag{17}$$

Renormalizing the wavevectors $(k_i, l_i) = c_-^{-1}(\tilde{k}_i, \tilde{l}_i)$ gives

$$\sqrt{1 + k_1^2 + l_1^2} + \sqrt{1 + k_2^2 + l_2^2} = -\frac{c_+}{c_-}(l_1 + l_2). \tag{18}$$

Condition of existence of resonance for negative $l_{1,2}$:

$$\frac{c_+}{c_-}(|l|_1 + |l|_2) > \sqrt{1 + |l|_1^2} + \sqrt{1 + |l|_2^2} \tag{19}$$

- satisfied for weak enough stratifications.

Analysis of resonance conditions: wave-mean

Resonance condition:

$$\sigma_{IG^-} = \sigma_{K^+}, \quad l_{IG^-} = l_{K^+} \equiv l \Rightarrow$$

$$c_+^2 l^2 = 1 + c_-^2 (k^2 + l^2), \tag{20}$$

which leads to the following expression for the x- component k of the wavenumber of the incoming IG wave:

$$k^2 = \left(\frac{c_+^2}{c_-^2} - 1 \right) l^2 - \frac{1}{c_-^2}. \tag{21}$$

For given stratification, i.e. $\frac{c_+^2}{c_-^2} > 1$, the resonance is always possible for sufficiently large negative l.

Evolution equation for the Kelvin wave envelopes

Wave-wave excitation

$$K^{\pm}_{T} + a_{\pm} c_{\pm} K^{\pm} K^{\pm}_{\xi^{\pm}} = \mathcal{S}, \tag{22}$$

where $\mathcal{S}$ - forcing by nonlinear interaction of IGW, harmonic in $\xi^{\pm}$.

Wave-mean excitation

$$K^{+}_{T} + C K^{+}_{\xi^{+}} + a_{+} c_{+} K^{+} K^{+}_{\xi^{+}} = \mathcal{S}, \tag{23}$$

C- velocity induced by the mean current (Doppler shift); $\mathcal{S}$ proportional to the amplitude of the IGW and harmonic in ξ^{+}

Here

$$a_{\pm} = \frac{1}{4}\left(\frac{1}{\sqrt{r}} \pm 1\right), \tag{24}$$

RSW equations with coast

Equations of motion:

$$\begin{aligned} u_t + uu_x + vu_y - fv + g\eta_x &= 0, \\ v_t + uv_x + vv_y + fu + g\eta_y &= 0, \\ \eta_t + ((\eta + h)u)_x + ((\eta + h)v)_y &= 0. \end{aligned} \quad (25)$$

Water depth $h(x)$ - 1d topography. Coast:

$$h|_{x=0} = 0, \quad h|_{x\to\infty} \to H = \text{const}. \quad (26)$$

Boundary conditions:

$$\eta|_{x\to 0} \text{ regular}, \quad \eta u|_{x\to 0} \to 0. \quad (27)$$

Linearized system

Linearized non-dimensional equations:

$$\begin{aligned} u_t - fv + \eta_x &= 0, \\ v_t + fu + \eta_y &= 0, \\ \eta_t + (hu)_x + (hv)_y &= 0, \end{aligned} \tag{28}$$

- Trapped waves:

$$(\eta,\ u,\ v)|_{x\to\infty} \to 0. \tag{29}$$

- Free (incident + reflected) wave:

$$(\eta,\ u,\ v)|_{x\to\infty} \propto e^{i(ly-\sigma t)} Re\left(A^+ e^{ikx} + A^- e^{-ikx}\right), \tag{30}$$

Linear waves

Wave equation:

Solutions of linearized system:

$$(u, v, \eta) = (iU, V, Z)\,(x) e^{i(ly - \sigma t)}, \Rightarrow \tag{31}$$

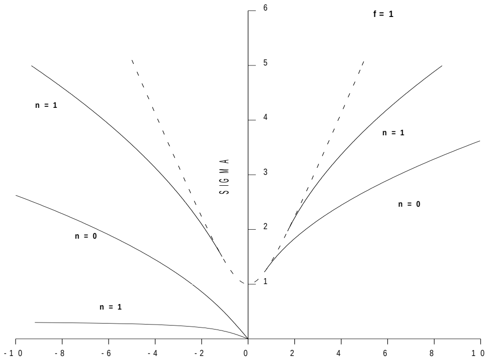
$$U = \frac{flZ - \sigma Z'}{\sigma^2 - f^2}, \quad V = \frac{\sigma lZ - fZ'}{\sigma^2 - f^2}, \quad ' \equiv \frac{d}{dx}. \tag{32}$$

Hence

$$(hZ')' + \left(\sigma^2 - f^2 - l^2 h - \frac{fl}{\sigma} h'\right) Z = 0, \tag{33}$$

an eigenproblem for eigenfunctions Z_n and eigenfrequencies σ_n. Solution below: Ball's model $h = 1 - e^{-x}$ (Ball, 1968); similar for any monotonous profile (Huthnance, 1975).

Dispersion diagram



Weakly nonlinear system

Nonlinear equations:

$$\begin{aligned} u_t - fv + \eta_x &= -\epsilon(uu_x + vu_y), \\ v_t + fu + \eta_y &= -\epsilon(uv_x + vv_y), \\ \eta_t + (hu)_x + (hv)_y &= -\epsilon\left((\eta u)_x + (\eta v)_y\right), \end{aligned} \tag{34}$$

$\epsilon = \frac{U}{\sqrt{gH}} << 1$ - Froude number.

Resonant triads

If

- lowest-order solution (u_0, v_0, η_0) is in the form:

$$\mathbf{W}_0 + \mathbf{W}_1 + \mathbf{W}_2,$$

where $\mathbf{W_i} = \mathbf{A_i}(\mathbf{iU_i}, \mathbf{V_i}, \mathbf{Z_i})\mathbf{e}^{\mathbf{i}\theta_\mathbf{i}}$, $\theta_\mathbf{i} = \mathbf{l_i y} - \sigma_\mathbf{i}\mathbf{t}$, $\mathbf{i} = 0, 1, 2$ - a pair of trapped waves, $\mathbf{i} = 1, 2$, and an incident/reflected wave $\mathbf{i} = 0$,

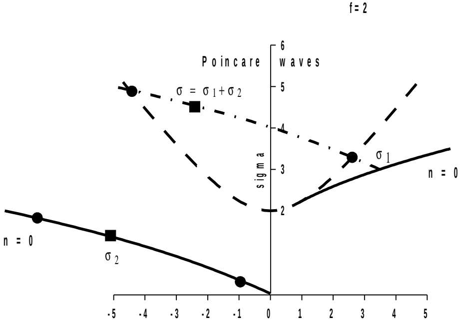
and

- synchronism conditions hold:

$$\mathbf{l}_0 = \mathbf{l}_1 + \mathbf{l}_2, \quad \sigma_0 = \sigma_1 + \sigma_2. \tag{35}$$

$\Rightarrow$ the amplitudes $\mathbf{A}_{1,2}$ grow exponentially in slow time $\mathbf{T}_2 = \epsilon^{-1}\mathbf{t}$.

Graphical solution for resonances



Modulation equations

Coupled generalized Landau equations:

$$
\begin{aligned}
\partial_{\mathbf{T}_2}\mathbf{A}_1 + \mathbf{i}\mathbf{M}_{11}\,|\mathbf{A}_1|^2\,\mathbf{A}_1 + \mathbf{i}\mathbf{M}_{12}\,|\mathbf{A}_2|^2\,\mathbf{A}_1 + \mathbf{i}\mathbf{M}_{02}\mathbf{A}_0\mathbf{A}_2^* &= 0 \\
\partial_{\mathbf{T}_2}\mathbf{A}_2 + \mathbf{i}\mathbf{M}_{21}\,|\mathbf{A}_1|^2\,\mathbf{A}_2 + \mathbf{i}\mathbf{M}_{22}\,|\mathbf{A}_2|^2\,\mathbf{A}_2 + \mathbf{i}\mathbf{M}_{01}\mathbf{A}_0\mathbf{A}_1^* &= 0
\end{aligned}
$$

$\mathbf{M_{ij}}$ - convolutions of zeroth and first orders for waves **i** and **j**, $\mathbf{M}_{02}\mathbf{M}_{01}^* > 0$ (condition for resonant growth).

General property:

Saturation $\equiv$ existence of attracting limit cycle, or attracting fixed point in $\mathbf{A}_1, \mathbf{A}_2$ - space (Details depend on the parameters of trapped waves).

Comments

- If spatial modulation taken into account - structure formation
- Earlier results (Miles, 1990) corrected and generalized for arbitrary incidence angle
- Details: Reznik & Zeitlin, JFM, 2011.

Conclusions

Several mechanisms of resonant excitation of trapped coastal waves exist, either leading to formation of coherent/dissipative structures in the coastal zones. Resonant pairs of free inertia-gravity waves, or resonances of free inertia-gravity waves with coastal currents produce coastal waves "from nothing"

Important for:

- Energy and momentum transport from the open ocean to the coast, and subsequent dissipation. A poorly explored route to dissipation in the ocean.
- Transport and mixing in the coastal zones.