Eddy generation by topographic transformation of coastal-trapped waves

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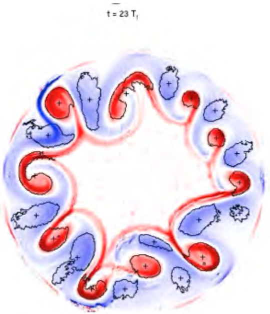
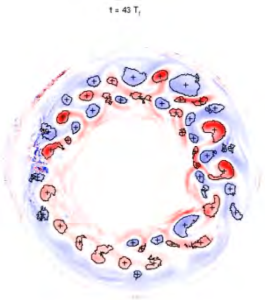
Eddies from instability



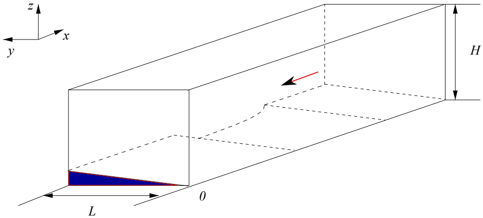
Flat bottom



Steep shelf

- Laboratory vorticity field (red: cyclonic/ blue: anticyclonic)
- Instability of a baroclinic current: R. Pennel, A. Stegner and K. Beranger (2012); F.G. Poulin, A. Stegner, M. Hernandez-Arencibia, A. Marrero-Diaz, and P. Sangra (2014)
- Another mechanism: same-signed eddies, no current, no instability. Just linear shelf waves.

The simplest geometry



- Channel with rigid side walls at $y = 0, L$, rigid lid at $z = H$.
- Channel floor is given by $h(x, y) = \alpha\beta(x)y$.
- α the bottom slope.
- $\beta(x)$ is $O(1)$ giving alongshore slope variation.
- A wave excited in $x > 0$ propagates along the wall $y = L$ and encounters a region of weaker slope.

The model

- Boussinesq with total density $\rho_0(z) + \rho(x, y, z, t)$, and pressure $p_0(z) + p(x, y, t)$, such that the equilibrium values ρ_0 and p_0 are in hydrostatic balance, i.e. $dp_0/dz = -\rho_0 g$.
- $\sigma = (-\rho/\rho*)g$ buoyancy acceleration
- $\mathcal{N}^2 = -(g/\rho*)d\rho_0/dz$ buoyancy frequency
- $\rho*$ is a constant reference density

Governing equations

Horizontal momentum: $$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho*}p_x,$$
$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho*}p_y,$$

Hydrostatic: $$p_z = \sigma,$$

Mass conservation: $$\frac{\mathrm{D}\sigma}{\mathrm{D}t} + \mathcal{N}^2 w = 0,$$

Continuity: $$u_x + v_y + w_z = 0,$$

Boundary conditions: $$v = 0, \quad \text{at} \quad y = 0, L,$$
$$w = 0, \quad \text{on} \quad z = H,$$
$$w = uh_x + vh_y \quad \text{on} \quad z = h(x, y),$$

i.e. $$w = u\alpha\beta'(x)y + v\alpha\beta(x), \quad \text{at} \quad z = \alpha\beta(x)y.$$

Here $\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + u\partial_x + v\partial_y + w\partial_z$.

Non-dimensionalisation

$$(x', y', z') = (x/L, y/L, z/H),$$
$$(u', v', w') = (u/U, v/U, wL/UH),$$
$$p' = p/\rho * fUL,$$
$$\sigma' = \sigma H/\rho * fUL,$$
$$t' = t/f\delta.$$

Two small parameters:

- $\delta = \alpha L/H$ the non-dimensional fractional depth change across the shelf
- $\mathrm{Ro} = U/fL$ Rossby number

One order unity parameter:
$B(z) = \mathcal{N}(z)H/fL$, a Burger number measuring stratification strength

Non-dimensional equations

$$
\begin{aligned}
\delta u_t + \mathrm{Ro}(uu_x + vu_y + wu_z) - v &= -p_x, \\
\delta v_t + \mathrm{Ro}(uv_x + vv_y + wv_z) + u &= -p_y, \\
\delta \sigma_t + \mathrm{Ro}(u\sigma_x + v\sigma_y) + B^2 w &= 0, \\
\sigma &= p_z, \\
u_x + v_y + w_z &= 0.
\end{aligned}
$$

Boundary conditions: no normal flow

$$
\begin{aligned}
v &= 0, \quad \text{at} \quad y = 0, 1, \\
w &= 0, \quad \text{at} \quad z = 1, \\
w &= \delta(uh_x + vh_y), \quad \text{on} \quad z = \delta h. \\
\text{i.e. } w &= \delta(u\beta' y + v\beta), \quad \text{at} \quad z = \delta\beta y.
\end{aligned}
$$

Geostrophic limit: $\delta \to 0$, $\mathrm{Ro} \to 0$ with δ/Ro fixed.
In fact can choose velocity scale U such that $\delta/\mathrm{Ro} = 1$.

Geostrophic limit

$$\frac{\mathrm{D}}{\mathrm{D}t}(p_{xx} + p_{yy} + (B^{-2}p_z)_z) = 0, \qquad \text{where } \frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \frac{\partial(p, .)}{\partial(x, y)},$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(p_z) = 0, \qquad \text{on } z = 1,$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(p_z + h) = 0, \qquad \text{on } z = 0.$$

For flow started from rest ($p \equiv 0$ at $t = 0$),

$$p_{xx} + p_{yy} + (B^{-2}p_z)_z = 0,$$

$$p_z = 0, \qquad \text{on } z = 1,$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(p_z + h) = 0, \qquad \text{on } z = 0.$$

Surface Geostrophic equations (SQG) (Johnson, 1977; 1978. Blumen 1978. Held *et al.*, 1995)

SQG

Write $\Sigma = \sigma|_{z=0} = p_z|_{z=0}$ and $P = p|_{z=0}$. Then

$$\frac{\mathrm{D}}{\mathrm{D}t}(\Sigma + h) = 0,$$

$$\text{where} \qquad \frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \frac{\partial(P, .)}{\partial(x, y)},$$

$$\text{and} \qquad P = \mathcal{L}\Sigma,$$

- Here $\mathcal{L}$ is a known linear operator $\mathcal{L}$: the Dirichlet-Neumann operator for the elliptic zero-PV interior equation.
- All dynamics is on the lower boundary and the interior flow is passive. (J., '77, '78)
- This is the full nonlinear problem: solved numerically (spectrally) here.

Linear waves

For *sufficiently small* disturbances the system (i.e. bottom b.c.) linearises to

$$
\begin{aligned}
\nabla^2 p + (B^{-2} p_z)_z &= 0, \\
p_x &= 0, \quad \text{at} \quad y = 0, 1, \\
p_z &= 0, \quad \text{at} \quad z = 1, \\
p_{zt} + B^2 (p_x h_y - p_y h_x) &= 0, \quad \text{on} \quad z = 0, \\
\text{i.e. } p_{zt} + B^2 (\beta(x) p_x - \beta'(x) y p_y) &= 0, \quad \text{on} \quad z = 0.
\end{aligned}
$$

If the bottom-slope β is constant then solutions are the simplest possible bottom-trapped shelf modes in the channel.

Slow alongshore variations

Consider waves of *fixed constant frequency* ω.
Let β vary slowly over a wavelength, i.e. $\beta = \beta(X)$, where $X = \varepsilon x$ ($\varepsilon \ll 1$). Look for a LG (1837) solution

$$p(X, y, z) \sim \exp\{(\mathrm{i}/\varepsilon) \int^X k(X')dX' + \mathrm{i}\omega t\} \sum_{j=0}^{\infty} \varepsilon^j \phi_j(X, y, z).$$

Leading order, B constant, gives Rhines (1970)

$$\begin{aligned} \phi_{0yy} + B^{-2}\phi_{0zz} - k^2\phi_0 = 0, & \\ \phi_0 = 0, \quad & \text{at} \quad y = 0, 1, \\ \phi_{0z} = 0, \quad & \text{at} \quad z = 1, \\ \omega\phi_{0z} = -kB^2\beta\phi_0, \quad & \text{on} \quad z = 0. \end{aligned}$$

Slowly-varying Rhines bottom-trapped modes

This gives the leading order solution

$$p(X, y, x) = F(X) \sin m\pi y \cosh \mu(z-1) \exp\{\mathrm{i}/\varepsilon \int^X k(X')dX'\} + O(\varepsilon)$$

- m is the (integral) cross-shelf wavenumber ($m = 1$ most important).
- $\mu(X) = B(k(X)^2 + m^2\pi^2)^{1/2}$ gives the (inverse of the) vertical scale.
- $F(X)$ slowly varying local amplitude (determined at next order in ε by conservation of wave energy flux, Green 1837).
- $k(X)$ slowly varying local along-shelf wavenumber.
- Recall the frequency ω is a fixed number.

Local dispersion relation

At this order $k(X)$ satisfies

$$\frac{\omega}{\beta(X)} = D(k(X)).$$

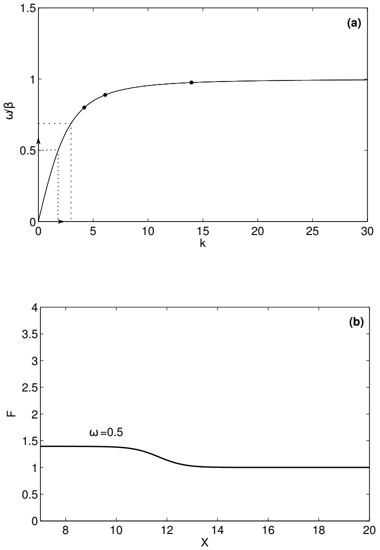
Here $$D(k) = \frac{B^2 k}{\mu \tanh \mu} = \frac{Bk}{\sqrt{k^2 + m^2\pi^2}\,\tanh(B\sqrt{k^2 + m^2\pi^2})}$$

is a function of k alone for each mode m.

- Relates the local wavenumber, $k(X)$, to the local cross-shelf slope, $\beta(X)$, parametrically.
- For $B \geq 1$ (moderate stratification) the dispersion curves are strictly monotonic increasing in k and so energy propagation is uni-directional,

$$\left.\frac{\partial \omega}{\partial k}\right|_X > 0.$$

Adiabatic transmission (Green 1837)

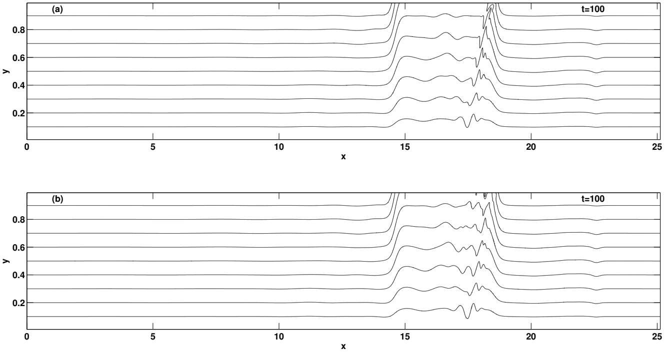


$\omega/\beta(X) = D(k)$ for $B = 1$.
Here $\omega = 0.5$.

- As $\beta(X)$ decreases, ω/β increases.
- Thus $k(X)$ increases, i.e. waves shorten.
- Graph slope, $\partial\omega/\partial k$, group velocity, decreases. Waves travel slower.
- Energy density, F, increases.

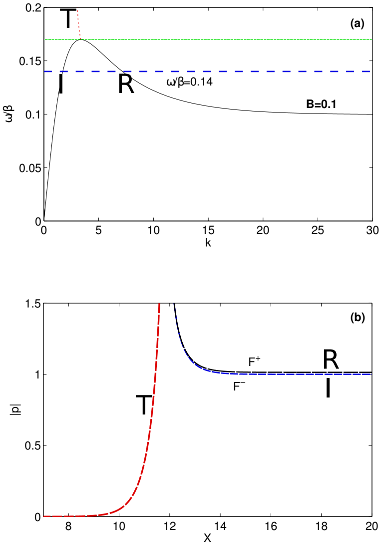
Bottom PV in the adiabatic transmission regime

Linear



Nonlinear (A=0.01)

One turning point (Harold Jeffreys 1923)

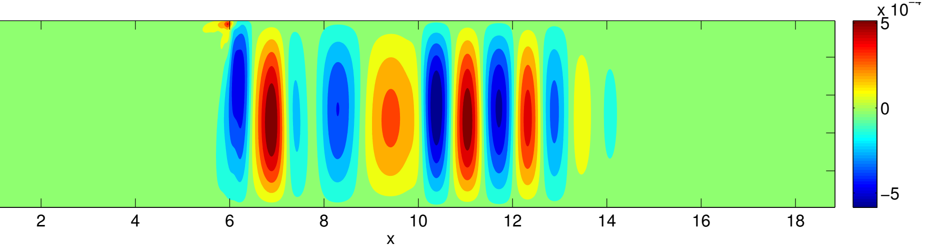


$B = 0.1$. Weak stratification.
$\omega = 0.14$.

- I – Incident long wave. Graph slope, $\partial\omega/\partial k$, group velocity, positive. Amplitude F^-.
- R – Reflected short wave. Graph slope, $\partial\omega/\partial k$, group velocity, negative. Amplitude F^+. Returns incident energy flux.
- T – Evanescent transmitted wave. Zero energy flux.
- Smoothed with Airy fn.

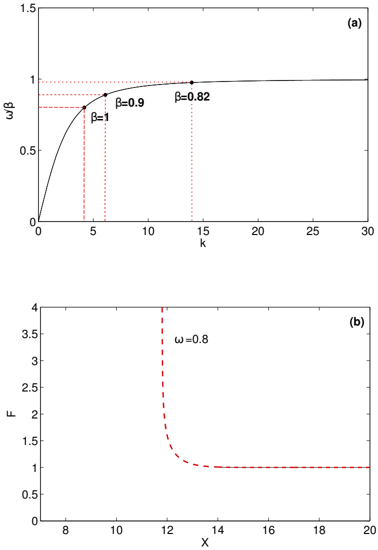
Linear reflection in the nonlinear equations

Quasi-barotropic flow (B=0.1, ω=0.16, A=0.01)



Perturbation surface PV, σ from a numerical solution SQG with nonlinear terms suppressed. (Not full SPV)

Singular absorption in the linear equations



Return to:- $\omega/\beta(X) = D(k)$ for $B = 1$ but here ω=0.8.

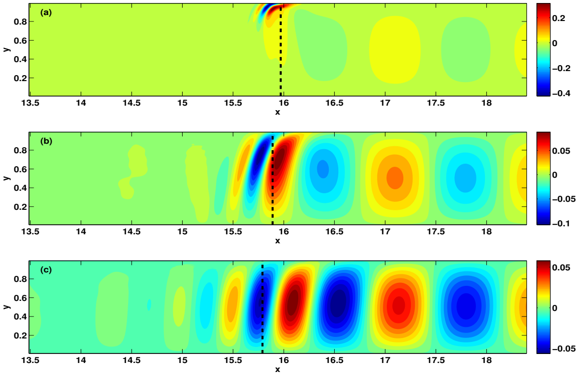
- As $\beta(X)$ decreases, ω/β increases.
- For $\beta(X) < 0.8$ there is no forward propagating wave.
- For this stratification there are no reflected waves.
- Energy accumulates at that X where the group velocity vanishes.
- The wavelength shrinks to zero.

Singular absorption in the linear equations

Perturbation surface PV, σ from a numerical solution of SQG with nonlinear terms suppressed. (Not full SPV)

Abrupt slope change Not LG

Smoother (LG better than might think- $k,\mu \to \infty$)



Right: constant energy flux. Left: zero energy flux

B=1, ω=0.8

Time dependence – Step-wall interaction

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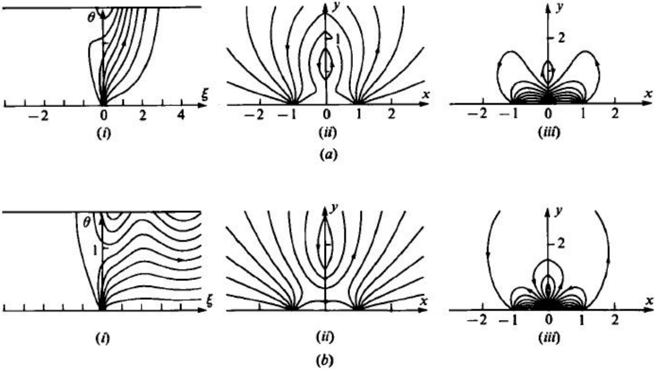


FIGURE 2. Streamline patterns at intermediate stages of the flow in figure 1. In both cases (*i*) gives the pattern in the (ξ, θ)-strip, (*ii*) the pattern in the original coordinates for $\gamma = 1$ and (*iii*) the pattern in the original coordinates for $\gamma = -1$. (*a*) $t = 2$; (*b*) $t = 10$. The scales are as in figure 1.

Linear, time-dependent (J. '85). Singularity remains.

Vertical variation – Internal Kelvin waves

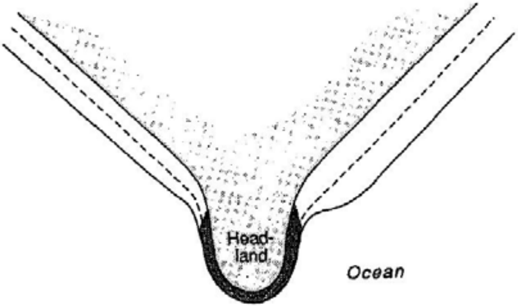
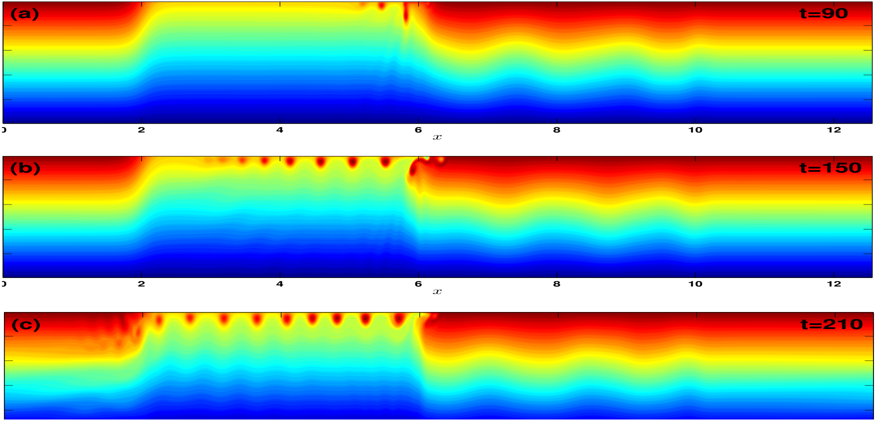


FIG. 13. Depth contours (in two dimensions) for a shelf interrupted by an arbitrarily large headland. When stratification is weak and frequencies are low ($\omega \ll B \ll 1$), the EKW carries the entire incident mass flux past the headland. Any remaining incident energy flux is transported without loss by IKWs in high velocity layers of thickness N_0H/f (shown shaded here) confined against vertical walls.

Linear, vertical variation (J. '91)

Nonlinear: Eddies in bottom PV

Full surface PV, $q = \sigma + h$ from a numerical solution SQG.



Nonlinear terms reinstated.

Closed contours of bottom PV = eddies. Single-signed.

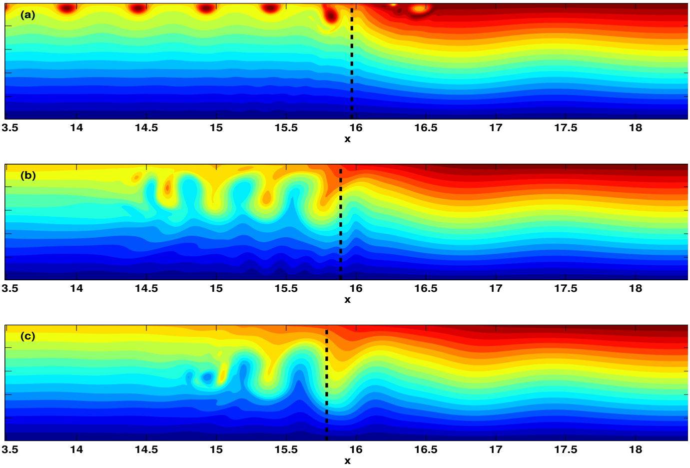
B=1, ω=0.8, A=0.01.

Eddies in bottom PV

Singular absorption regime in nonlinear equations

Abrupt slope change Not LG

Smoother (LG better than might think- $k, \mu \to \infty$)



Numerical solution for SPV with nonlinear terms reinstated.
Not perturb SPV., *sufficiently small?* (B=1, ω=0.8)

Surface PV density and flux

- The governing equation can be written

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{u}) = 0, \quad \text{on} \quad z = 0,$$

  where $q = \sigma + h$ is the surface PV.
- Integrating across the shelf gives the conservation of along-shelf SPV as

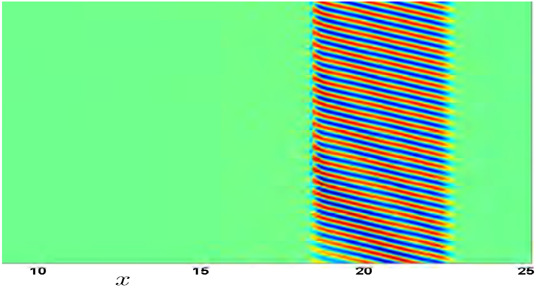
$$\frac{\partial Q}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0,$$

  where

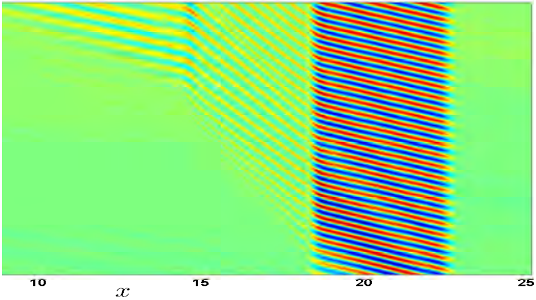
$$Q(x,t) = \int_0^1 q\, dy, \qquad \mathcal{F}(x,t) = \int_0^1 uq\, dy.$$

  are the cross-shelf-averaged SPV density and cross-shelf-averaged SPV flux.
- The SPV evolution can thus be shown in a Hovmöller diagram, plotting the flux $\mathcal{F}$ as a function of x and t.

Hovmöller diagram for bottom PV flux



Singular absorption regime.
(B=1, ω=0.8, A=0.01)
(a) linear



(b) nonlinear

Conclusion

- At sufficiently strong stratifications and shelf slopes no short counter-propagating waves exist.
- Low frequency incoming shelf wave energy can be transmitted through adiabatic transformation (LG).
- Higher frequency shelf waves can be transmitted as Internal Kelvin Waves or as eddies or nonlinear waves, depending on their amplitude and the rapidity of change of bottom slope.

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