LETG UMR 6554 CNRS, Rennes 2 Univ., May 22th, 2017

Information Fusion with Belief Functions and Multi-Criteria Decision-Making Support

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return on innovation

Outline

1 - Information Fusion with Belief functions

2 - Decision-Making with Belief functions

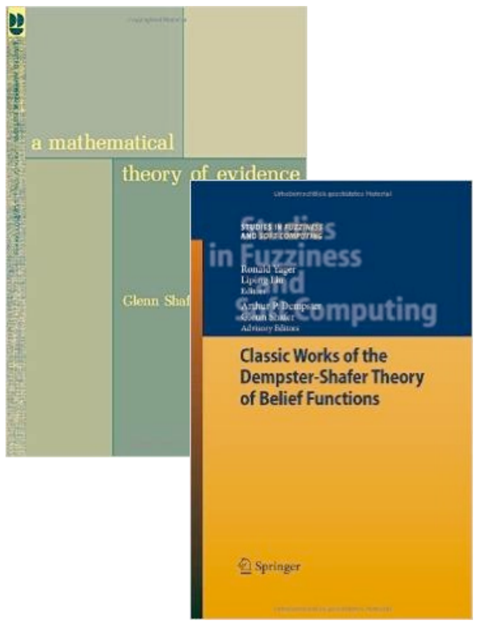
3 - Multi-Criteria Decision-Making Support

Part 1

Information Fusion with Belief Functions

Main references

Some references on Dempster-Shafer Theory (DST)



G. Shafer, A mathematical theory of evidence, 1976.

R. Yager, L. Liu, Classic Works of the Dempster-Shafer Theory of Belief Functions, 2008.

http://www.glennshafer.com/books/amte.html

Some references on Dezert-Smarandache Theory (DSmT)

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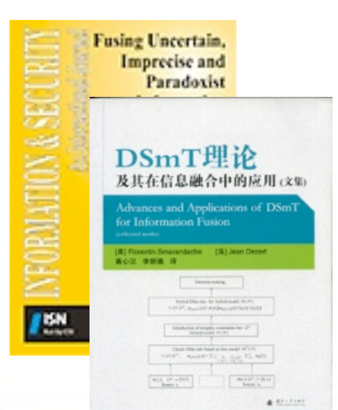


Toolboxes: http://bfasp.iutlan.univ-rennes1.fr/wiki/index.php/Toolboxes
http://martin.iutlan.univ-rennes1.fr/Doc/GeneralBeliefFunctionsFramework.tar

F. Smarandache, J. Dezert (Eds), Advances and applications of DSmT for information fusion, Vols. 1-4, 2004, 2006, 2009 & 2015.

http://www.onera.fr/fr/staff/jean-dezert
http://www.smarandache.com/DSmT.htm

Limitations of probabilities

They do not account for partial/incomplete knowledge.

They deal generally with information drawn from generic knowledge based either on population of items, laws of physics, common sense, ...

They capture only one aspect of the uncertainty
(the randomness, i.e. the variability through repeated measurements).

They can’t distinguish between uncertainty due to variability, and uncertainty due to the incompleteness/lack of knowledge (epistemic uncertainty).

Variability is related with precisely observed random observations

Incompleteness/non specificity is related with missing/partial information

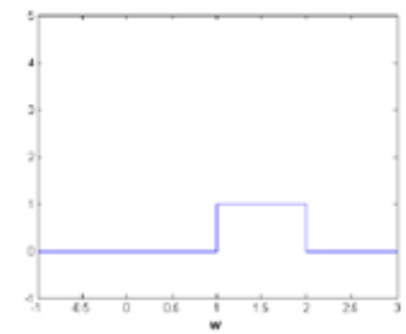
Why uniform pdf is not satisfactory

Limitation of uniform prior pdf to model the ignorance

Consider a random variable W taking its value w in [1,2], and the random variable V=1/W which obviously takes its value v=1/w in [0.5,1].

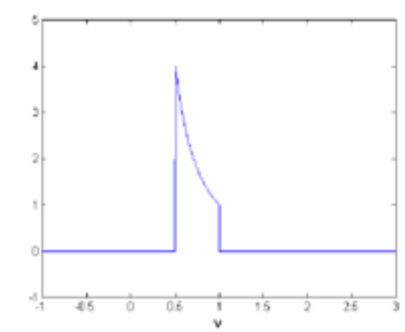
To model ignorance of value of W, it is usually assumed **uniform prior pdf**.

$$W \sim u([1,2)]) \Leftrightarrow P(W \leq w) = \begin{cases} 0 & \text{if} \quad w < 1 \\ w-1 & \text{if} \quad 1 \leq w \leq 2 \\ 1 & \text{if} \quad w > 2 \end{cases} \qquad p_W(w) = \frac{\partial}{\partial w} P(W \leq w) = \begin{cases} 0 & \text{if} \quad w \notin [1,2] \\ 1 & \text{if} \quad w \in [1,2] \end{cases}$$



By doing so, however we get **Non-uniform prior** pdf for V=1/W.

$$\begin{aligned} P(V \leq v) &= P(\frac{1}{W} \leq v) = P(W \geq \frac{1}{v}) = 1 - P(W < \frac{1}{v}) \\ &= \begin{cases} 1 & \text{if} \quad \frac{1}{v} < 1 \\ 2 - \frac{1}{v} & \text{if} \quad \frac{1}{v} \in [1,2] \\ 0 & \text{if} \quad \frac{1}{v} > 2 \end{cases} \end{aligned} \qquad p_V(v) = \frac{\partial}{\partial v} P(V \leq v) = \begin{cases} 0 & \text{if} \quad v \notin [\frac{1}{2}, 1] \\ \frac{1}{v^2} & \text{if} \quad v \in [\frac{1}{2}, 1] \end{cases}$$



which is not satisfactory because, we are a priori fully ignorant on the true value of W as well as of 1/W !!! So the choice of uniform pdf **does not model properly** our prior full ignorance of values w and v.

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Frame of discernment and Shafers' model

Paradigm shift with Belief Functions (BF)

Beliefs often are related with singular event and are **not necessarily related with statistical data** and generic knowledge, but with singular evidence. BF are well adapted for modeling partial knowledge.

Frame of discernment (FoD) $\Theta = \{\theta_i, i = 1, \ldots, n\}$

Shafer's model Close world assumption with exclusivity of elements

Power-set $\mathcal{P}(\Theta) \triangleq 2^{\Theta}$

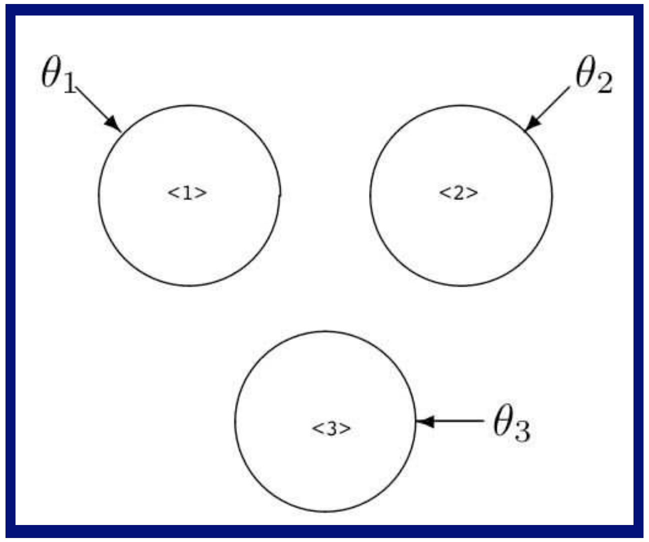
Any subset A of the FoD corresponds to the proposition

$$\mathcal{P}_\theta(A) \triangleq The\ true\ value\ of\ \theta\ is\ in\ a\ subset\ A\ of\ \Theta.$$

There is equivalence between operators on sets and logical operators

Power set example

$$\Theta = \{\theta_1, \theta_2, \theta_3\} \Rightarrow$$



Impossibility — **partial ignorances** — **full ignorance**

$$2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_2 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}$$

$$|2^\Theta| = 2^3 = 8$$

Belief functions

Basic belief assignment (BBA) $m(.) : 2^{\Theta} \rightarrow [0,1]$

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in 2^{\Theta}} m(A) = 1$$

Focal element A: iff m(A)>0

Vacuous BBA $\forall A \neq \Theta, m_v(A) = 0 \text{ and } m_v(\Theta) = 1$

Credibility $$\mathrm{Bel}(A) = \sum_{B \in 2^{\Theta}, B \subseteq A} m(B)$$

Total mass of subsets implying A

Plausibility $$\mathrm{Pl}(A) = \sum_{B \in 2^{\Theta}, B \cap A \neq \emptyset} m(B)$$

Total mass of subsets intersecting A

In general, $0 \leq \mathrm{Bel}(A) \leq \mathrm{Pl}(A) \leq 1$

«Bayesian» BBA Focal elements are singletons Bel(A)=Pl(A)=P(A)

[G. Shafer, A mathematical theory of evidence, 1976.]

Shafer's discounting

Discounting a source of evidence (Shafer's reliability discounting)

$$\begin{cases} m(A) \\ m(\Theta) \end{cases} \rightarrow \begin{cases} m'(A) = \alpha \cdot m(A) & \forall A \neq \Theta \\ m'(\Theta) = (1-\alpha) + \alpha \cdot m(\Theta) \end{cases}$$

$\alpha = 1$ means no discounting (full reliability of the source)
$\alpha = 0$ means total discounting (full unreliable/ignorant source)

To be used if one has a good estimation of the reliability factor of the source based on experiments and ground truth.

Other discounting techniques

- **Contextual discounting** [Denœux et al. 2005, 2006]
- **Importance discounting**

[Smarandache F., Dezert J., Tacnet J.-M., Fusion of sources of evidence with different importances and reliabilities, Proc. of Fusion 2010.]

Dempster's rule of combination

Combination of 2 sources of evidence

$$m_{DS}(\emptyset) \triangleq 0$$

$$\forall X \neq \emptyset \in 2^\Theta \qquad \boxed{m_{DS}(X) = [m_1 \oplus m_2](X) = \frac{m_{12}(X)}{1 - K_{12}}}$$

$$\underbrace{m_{12}(X) \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)}_{\text{Conjunctive fusion}} \qquad \underbrace{K_{12} \triangleq m_{12}(\emptyset) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)}_{\text{Conflict level}}$$

DS rule = Normalized conjunctive rule

[G. Shafer, A mathematical theory of evidence, 1976.]

Properties of DS rule and conditioning

- Properties: extension to $n > 2$ sources; associativity; commutativity; neutrality of vacuous bba $[m \oplus m_v](.) = m(.)$

Because of its properties, DS rule is very appealing.

- Conditioning: $m(.)$ combined with $m_Z(.)$ focused on Z (i.e. $m_Z(Z) = 1$) with DS rule yields $m(X|Z) = [m \oplus m_Z](X) = [m_Z \oplus m](X)$ and $Pl(X|Z) = Pl(X \cap Z)/Pl(Z)$ (similar to Conditioning rule for probas).

Because of this, DS rule has often been wrongly interpreted as a generalization of Bayes rule. See the following papers

[A. Brodzik, R. Enders, A case of combination of evidence in the Dempster-Shafer theory inconsistent with evaluation of probabilities, 2011.]
https://arxiv.org/pdf/1107.0082.pdf

[J. Heendeni et al., A Generalization of Bayesian Inference in the Dempster-Shafer Belief Theoretic Framework, in Proc. Fusion 2016.]

Drawbacks of DS rule

DS rule is mathematically not defined when conlict is total (K=1).

DS rule doesn't behave well not only in high conflicting case [Zadeh 1979], but even **in low** conflicting case.

[Dezert J., Wang P., Tchamova A., On The Validity of Dempster-Shafer Theory, Proc. of Fusion 2012.]

DS rule is **not** a generalization of Bayes rule because it is incompatible with Bayes rule when the prior is not uniform, nor vacuous.

[A. Brodzik, R. Enders, A case of combination of evidence in the Dempster-Shafer theory inconsistent with evaluation of probabilities, 2011.]
https://arxiv.org/pdf/1107.0082.pdf

[Dezert J., Tchamova A., Han D., Tacnet J.-M., Why Dempster's fusion rule is not a generalization of Bayes fusion rule, Proc. of Fusion 2013.]

[Dezert J., Tchamova A., On the validity of Dempster's fusion rule and its interpretation as a generalization of Bayesian fusion rule, Int. J. of Intelligent Systems, Vol. 29 (3), 2014.]

Zadeh's example (1979)

High conflict case $\Theta = \{\theta_1 = \text{Tumor}, \theta_2 = \text{Meningitis}, \theta_3 = \text{Concussion}\}$ Shafer's model

Bayesian BBAs

$$m_1(\theta_1) = 1 - e_1 \qquad m_1(\theta_2) = 0 \qquad m_1(\theta_3) = e_1$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 1 - e_2 \qquad m_2(\theta_3) = e_2$$

$$k_{12} = (1-e_1)(1-e_2) + (1-e_1)e_2 + e_1(1-e_2) = 1 - e_1e_2$$

If $e_1 = 0.1$ and $e_2 = 0.1$, then $k_{12} = 1 - 0.01 = 0.99$ **(high conf.)**

DS fusion result

$$m(\theta_3) = \frac{e_1e_2}{(1-e_1)\cdot 0 + 0\cdot(1-e_2) + e_1e_2} = 1$$

DS rule provides same result whatever the positive values of e_1 and e_2 are !!!

DS is not numerically robust to slight input changes. If we take very small non zero values instead of zeros, we get: $m(\theta_1) = m(\theta_2) \approx 0.5 \quad m(\theta_3) \approx 0$

[J. Dezert, A. Tchamova, D. Han, A Real Z-box Experiment for Testing Zadeh's Example, in Proc. of Fusion 2015.]

Modified Zadeh's example

Low conflict case $\Theta = \{\theta_1 = \text{Tumor}, \theta_2 = \text{Meningitis}, \theta_3 = \text{Concussion}\}$ Shafer's model

Bayesian BBAs

$$m_1(\theta_1) = 0.99 \quad m_1(\theta_2) = 0.01 \quad m_1(\theta_3) = 0$$
$$m_2(\theta_1) = 0.99 \quad m_2(\theta_2) = 0 \quad m_2(\theta_3) = 0.01$$

$$k_{12} = (0.99 \cdot 0.01) + (0.01 \cdot 0.99) + (0.01 \cdot 0.01) = 0.0199$$

DS fusion result $$m(\theta_1) = \frac{m_1(\theta_1)m_2(\theta_1)}{1 - k_{12}} = \frac{0.99 \cdot 0.99}{1 - 0.0199} = \frac{0.9801}{0.9801} = 1$$

One gets complete support (i.e. certainty) for the diagnosis of a brain tumor !!!

This result is counter-intuitive since there is not a full agreement (there is a conflict) between the two sources, and the existence of non-zero probas for other diagnoses implies less than complete support. So we should expect in this case to get by the fusion

$$m(\theta_1) < 1$$

Dezert-Tchamova example (2011)

Low conflict case $\Theta = \{\theta_1, \theta_2, \theta_3\}$ Shafer's model

Non-Bayesian BBAs

Focal elem. \ bba's	$m_1(.) \neq m_V(.)$	$m_2(.) \neq m_V(.)$
A	a	0
$A \cup B$	$1-a$	b_1
C	0	$1-b_1-b_2$
$A \cup B \cup C$	0	b_2

Conjunctive fusion

$$m_{12}(A) = m_1(A)m_2(A \cup B) + m_1(A)m_2(A \cup B \cup C) = a(b_1 + b_2)$$

$$m_{12}(A \cup B) = m_1(A \cup B)m_2(A \cup B) + m_1(A \cup B)m_2(A \cup B \cup C) = (1-a)(b_1 + b_2)$$

Conflicting mass

$$\begin{aligned} K_{12} = m_{12}(\emptyset) &= m_1(A)m_2(C) + m_1(A \cup B)m_2(C) \\ &= a(1-b_1-b_2) + (1-a)(1-b_1-b_2) \\ &= 1-b_1-b_2 \end{aligned}$$

chosen as low as we want.

[Dezert J., Wang P., Tchamova A., On The Validity of Dempster-Shafer Theory, in Proc. of Fusion 2012.]

Dezert-Tchamova example (cont'd)

$$m_{12}(A) = m_1(A)m_2(A \cup B) + m_1(A)m_2(A \cup B \cup C) = a(b_1 + b_2)$$

$$m_{12}(A \cup B) = m_1(A \cup B)m_2(A \cup B) + m_1(A \cup B)m_2(A \cup B \cup C) = (1-a)(b_1 + b_2)$$

After normalization by $1 - K_{12} = b_1 + b_2$ one gets, with DS rule

$$m_{DS}(A) = \frac{m_{12}(A)}{1 - K_{12}} = \frac{a(b_1 + b_2)}{b_1 + b_2} = a = m_1(A)$$

$$m_{DS}(A \cup B) = \frac{m_{12}(A \cup B)}{1 - K_{12}} = \frac{(1-a)(b_1 + b_2)}{b_1 + b_2} = 1 - a = m_1(A \cup B)$$

- $m_{DS}(.) = [m_1 \oplus m_2](.) = m_1(.)$ even if $m_2(.) \neq m_v(.)$
- The informative source $m_2(.)$ doesn't count $\Rightarrow$ Dictatorial power of DS rule
- The level of conflict K_{12} doesn't matter in the result.

This Dempster-Shafer fusion result is **very** counter intuitive

Incompatibility of DS rule with Bayes rule

Bayesian BBAs

$$\begin{cases} m_1(x_1) = P(X = x_1|Z_1) = 0.2 \\ m_1(x_2) = P(X = x_2|Z_1) = 0.3 \\ m_1(x_3) = P(X = x_3|Z_1) = 0.5 \end{cases} \quad \text{and} \quad \begin{cases} m_2(x_1) = P(X = x_1|Z_2) = 0.5 \\ m_2(x_2) = P(X = x_2|Z_2) = 0.1 \\ m_2(x_3) = P(X = x_3|Z_2) = 0.4 \end{cases}$$

Using **informative prior** bba/pmf:

$$\begin{cases} m_0(x_1) = P(X = x_1) = 0.6 \\ m_0(x_2) = P(X = x_2) = 0.3 \\ m_0(x_3) = P(X = x_3) = 0.1 \end{cases}$$

Bayes rule

$$\begin{cases} P(x_1|Z_1 \cap Z_2) & = \frac{0.2 \cdot 0.5/0.6}{2.2667} = \frac{0.1667}{2.2667} \approx 0.0735 \\ P(x_2|Z_1 \cap Z_2) & = \frac{0.3 \cdot 0.1/0.3}{2.2667} = \frac{0.1000}{2.2667} \approx 0.0441 \\ P(x_3|Z_1 \cap Z_2) & = \frac{0.5 \cdot 0.4/0.1}{2.2667} = \frac{2.0000}{2.2667} \approx 0.8824 \end{cases}$$

DS rule

$$\begin{cases} m_{DS}(x_1) & = \frac{1}{1-0.9110} \cdot 0.2 \cdot 0.5 \cdot 0.6 = \frac{0.060}{0.089} \approx 0.6742 \\ m_{DS}(x_2) & = \frac{1}{1-0.9110} \cdot 0.3 \cdot 0.1 \cdot 0.3 = \frac{0.009}{0.089} \approx 0.1011 \\ m_{DS}(x_3) & = \frac{1}{1-0.9110} \cdot 0.5 \cdot 0.4 \cdot 0.1 = \frac{0.020}{0.089} \approx 0.2247 \end{cases}$$

DS rule is incompatible with Bayes rule in general. [Dezert/Tchamova/Han/Tacnet 2013]

DS rule is compatible with Bayes rule **only** if the prior is uniform or vacuous.

Why going beyond DST

Major innovations of DST

- Important paradigm shift for modeling uncertainty
- New appealing mathematical formalism of (quantitative) belief functions
- New combination rule for belief functions (DS rule)

... but BF and DST have never been fully accepted by a part of scientific community and statisticians mainly because

- Independence between sources of evidence has never been well defined
- Doubts on the validity of DS rule (normalization is controversial)
- Good experimental protocol to validate DST and DS rule is lacking

See Zadeh 1979,Yager 1983, Lemmer 1985, Dubois 1986, Pearl 1988,Voorbraak 1991, Wang 1994, Walley 1996, Fixsen et al. 1997, Gelman 2006, Dezert & al. 2012, etc

Recommendation

What we have shown

- the dictatorial power of DS rule to fuse equi-reliable sources of evidence.
- the conflict (high or low) can be totally ignored through DS rule.
- the problem of validity of DST is not due to conflict level, but the absolute truth interpretation of proposition by Shafer for each source.
- We have proved a logical contradiction in the foundations of DST in

  [Dezert J., Tchamova A., On the validity of Dempster's fusion rule and its interpretation as a generalization of Bayesian fusion rule, Int. J. of Intelligent Systems, Vol. 29 (3), March 2014.]

Recommendation

BF are mathematically appealing and well defined, but don't use DS rule to combine them, even in low conflicting situations.

Reducing troubles with DST

Some tricks to reduce troubles with DS rule

1) Apply ad-hoc thresholdings on the conflict to accept (or reject) DS result.

2) Modify input BBAs, or apply discounting techniques on sources.
   - How to be sure that no problem will occur with DS rule after discounting ?
   - How to discount sources when no statistical data is available ?

3) Mix the two previous strategies.

How to better prevent troubles ?

Switch tobetter (more efficient) techniques to fusion vague, uncertain, imprecise, conflicting quantitative and qualitative information fusion for static or dynamic problematics.

This is what Dezert-Smarandache Theory (**DSmT**) proposes.

DSmT versus DST in short

Shafer's interpretation: A source can provide absolute truth from partial knowledge, observation, experience, ...

Such interpretation yields a logical contradiction in DST foundations.

Our interpretation: A source can provide only a relative truth from partial knowledge, observation, experience, ...

This new interpretation makes differences in the way to process belief functions.

DSmT in short

What is DSmT (developed between 2002 and 2015)

It is a natural extension of the belief function framework to work with
- different models for the frame (not only Shafer's model)
- with (possibly imprecise) quantitative belief functions
- with qualitative belief functions (expressed as labels)
- new PCR rules of combination, and conditioning
- new probabilistic transformation and method for decision-making support

Why to use DSmT

- provides better results in fusion applications than DST
- deals with static and dynamic frames in a same general framework
- can cover broader fields of applications (because of more flexibility)

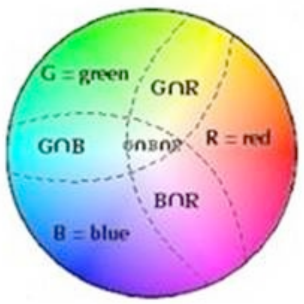
Drawback of DSmT

- its higher complexity (from theoretical and implementation standpoints)

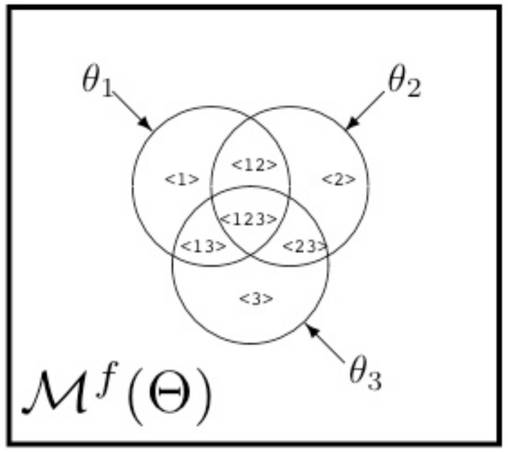
DSmT models

Free DSm model

No constraint on elements of the frame

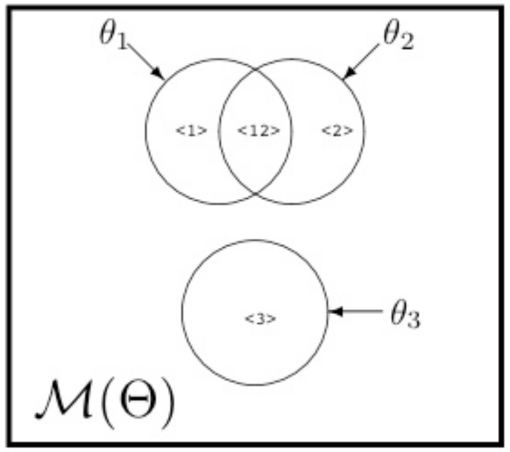


Parts can have vague boundaries



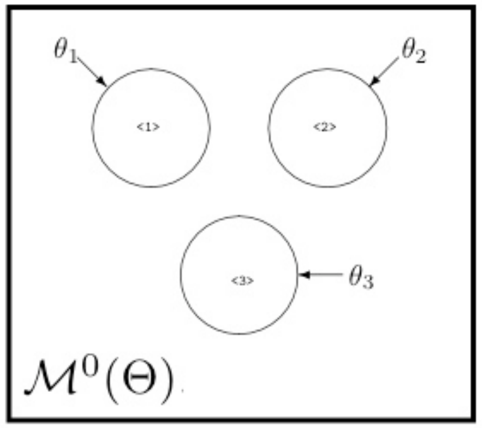
Hybrid DSm model

We introduce integrity constraints into the free DSm model.



Shafer's model = specific hybrid model

All exhaustive elements of the frame are known to be truly exclusive (i.e. a «refinement» is implicitly done)



Parts have precise boundaries

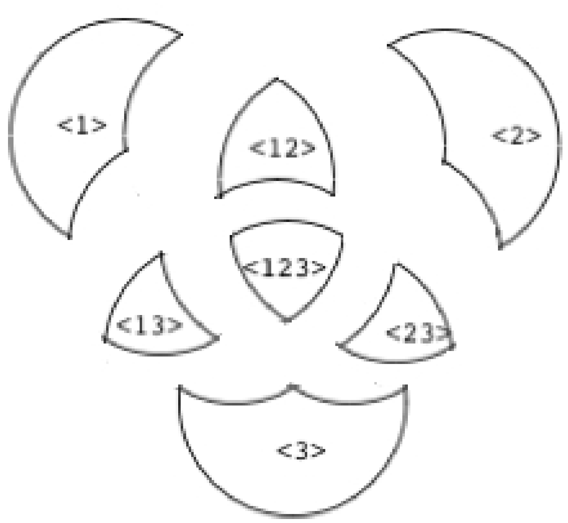
Fusion spaces

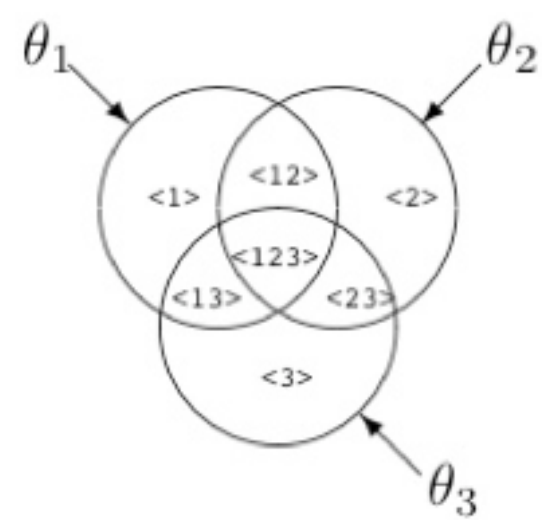
FoD $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}.$ Finite set of exhaustive elements (discrete/continuous/fuzzy/relative concepts)

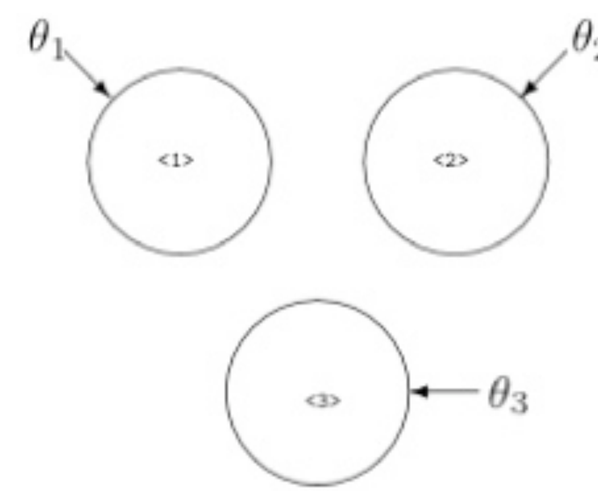
Fusion spaces

Power sets, Hyper-power set (Dedekind's lattice) and Super-power sets

$$|2^{\Theta_{ref}} = S^{\Theta} \triangleq (\Theta, \cup, \cap, c(.))| > |D^{\Theta} = (\Theta, \cup, \cap)| > |2^{\Theta} = (\Theta, \cup)|$$







Super-power set = power set of the refined frame

Hyper-power set

Generation of hyper-power sets

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$
2. $\forall A \in D^\Theta, B \in D^\Theta, (A \cup B) \in D^\Theta, (A \cap B) \in D^\Theta$
3. No other elements belong to D^Θ, except those, obtained by using rules 1 or 2.

Hyper-power set reduces to classical power set for the Shafer's model (when all elements are exclusive)

The cardinality of hyper-power sets follows Dedekind's numbers sequence when the size of the frame increases.

Example (FoD with n=3 elements)

$$\Theta = \{\theta_1, \theta_2, \theta_3\}$$

d(n=3)=19

$\alpha_0 \triangleq \emptyset$	$\alpha_4 \triangleq \theta_2 \cap \theta_3$	$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3$	$\alpha_{16} \triangleq \theta_1 \cup \theta_3$
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3$	$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3$	$\alpha_9 \triangleq \theta_1$	$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2$	$\alpha_{17} \triangleq \theta_2 \cup \theta_3$
$\alpha_2 \triangleq \theta_1 \cap \theta_2$	$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2$	$\alpha_{10} \triangleq \theta_2$	$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1$	$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3$
$\alpha_3 \triangleq \theta_1 \cap \theta_3$	$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1$	$\alpha_{11} \triangleq \theta_3$	$\alpha_{15} \triangleq \theta_1 \cup \theta_2$	

Belief functions in DSmT

$$m(.) : G^{\Theta} \to [0,1] \qquad m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in G^{\Theta}} m(A) = 1$$

$$\mathrm{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in G^{\Theta}}} m(B) \qquad \text{and} \qquad \mathrm{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in G^{\Theta}}} m(B)$$

where G^{Θ} is the fusion space (i.e. 2^{Θ}, D^{Θ}, or $S^{\Theta} = 2^{\Theta_{refined}}$)

One can also define **qualitative** BBA's (using labels), and **imprecise** admissible (quantitative or qualitative) BBA's - see [DSmTBooks]

PCR rules of combinations

1 - Apply the conjunctive rule

2 - Calculate the total or partial conflicting masses

3 - Redistribute the (total or partial) conflicting mass proportionally on non-empty sets according to the integrity constraints one has for the FoD

The proportional conflict redistribution (PCR) can be done in many ways.

- PCR rule #5 (PCR5) proposed by Smarandache & Dezert [DSmTBook3]
- PCR rule #6 (PCR6) proposed by Martin & Osswald [DSmTBook3]

$$m_{PCR5/6}(X) = \sum_{\substack{X_1,X_2\in 2^\Theta \\ X_1\cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{Y\in 2^\Theta\setminus\{X\} \\ X\cap Y=\emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}]$$

PCR5 = PCR6 for combining 2 sources

PCR5 ≠ PCR6 for combining *s>2* sources

PCR6 is better than PCR5

Example for PCR5/6 rules

Combining two BBAs with PCR5/6 rules

See [DSmTBooks] for general formulas

$$m_{PCR5/6}(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{Y \in 2^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}]$$

Example

$\Theta = \{A, B\}$

	A	B	$A \cup B$
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.2	0.3	0.5
$m_{12}(.)$	0.44	0.27	0.05

$$m_{12}(A \cap B = \emptyset) = m_1(A)m_2(B) + m_1(B)m_2(A)$$
$$= 0.18 + 0.06 = 0.24$$

$$x_1/0.6 = y_1/0.3 = (x_1 + y_1)/(0.6 + 0.3) = 0.18/0.9 = 0.2 \longrightarrow \begin{cases} x_1 = 0.6 \cdot 0.2 = 0.12 \\ y_1 = 0.3 \cdot 0.2 = 0.06 \end{cases}$$

$$x_2/0.2 = y_2/0.3 = (x_2 + y_2)/(0.2 + 0.3) = 0.06/0.5 = 0.12 \longrightarrow \begin{cases} x_2 = 0.2 \cdot 0.12 = 0.024 \\ y_2 = 0.3 \cdot 0.12 = 0.036 \end{cases}$$

$$m_{PCR5/6}(A) = 0.44 + \boxed{0.12} + \boxed{0.024} = 0.584$$
$$m_{PCR5/6}(B) = 0.27 + \boxed{0.06} + \boxed{0.036} = 0.366$$
$$m_{PCR5/6}(A \cup B) = 0.05 + 0 = 0.05$$

With Dempster's rule
$m_{DS}(A) \approx 0.579$
$m_{DS}(B) \approx 0.355$
$m_{DS}(A \cup B) \approx 0.066$

The mass put on ignorance with PCR5/6 is lower than with DST

Difference between PCR5 and PCR6

Example $\Theta = \{A, B\}$ Shafer's model

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$
$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$
$$m_3(A) = 0.7 \quad m_3(B) = 0.1 \quad m_3(A \cup B) = 0.2$$

Let's consider the partial conflicting mass.

$$m_1(A)m_2(B)m_3(B) = 0.6 \cdot 0.3 \cdot 0.1 = 0.018$$

With PCR5, one takes $$\frac{x_A^{PCR5}}{m_1(A)} = \frac{x_B^{PCR5}}{m_2(B)m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + m_2(B)m_3(B)}$$

$$\frac{x_A^{PCR5}}{0.6} = \frac{x_B^{PCR5}}{0.03} = \frac{0.018}{0.6 + 0.03} \approx 0.02857 \longrightarrow \begin{cases} x_A^{PCR5} = 0.60 \cdot 0.02857 \approx 0.01714 \\ x_B^{PCR5} = 0.03 \cdot 0.02857 \approx 0.00086 \end{cases}$$

With PCR6, one takes $$\frac{x_A^{PCR6}}{m_1(A)} = \frac{x_B^{PCR6}}{m_2(B) + m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + (m_2(B) + m_3(B))}$$

$$\frac{x_A^{PCR6}}{0.6} = \frac{x_B^{PCR6}}{0.3 + 0.1} = \frac{0.018}{0.6 + (0.3 + 0.1)} = 0.018 \longrightarrow \begin{cases} x_A^{PCR6} = 0.6 \cdot 0.018 = 0.0108 \\ x_B^{PCR6} = (0.3 + 0.1) \cdot 0.018 = 0.0072 \end{cases}$$

PCR6 result is more stable than PCR5 result for decision making, and PCR6 is consitent with frequentist proba estimate.

Advantages and drawbacks of PCR rules

Advantages PCR5/6 rules work with any conflict, and outperfom DS rule.

Drawbacks Complexity, non-associativity

Why PCR6 is considered better than PCR5 and DS rule

PCR6 can be used to estimate correctly frequentist probas in random binary experiment. DS and PCR5 do not work.

[Smarandache F., Dezert J., On the consistency of PCR6 with the averaging rule and its application to probability estimation, Proc. of Fusion 2013.]

Complexity of DSmT

Complexity of BF

$$(|2^\Theta| = 2^n) < (|D^\Theta| = d(n)) < (|2^{\Theta_{ref}}| = 2^{2^n - 1})$$

$\|\Theta\| = n$	$\|2^\Theta\| = 2^n$	$\|D^\Theta\| = d(n)$	$\|2^{\Theta_{ref}}\| = 2^{2^n-1}$
2	4	5	$2^3 = 8$
3	8	19	$2^7 = 128$
4	16	167	$2^{15} = 32768$
5	32	7580	$2^{31} = 2147483648$

How to reduce complexity for combining BF

Approximate BBA by simpler ones

Implement fusion rules with sampling techniques [DSmT Book 3,Chap6]

Use simpler fusion rules

Probabilistic transformations

Approximate a BBA by a simpler one (probabilistic transforms)

Simplest method keeps only singletons as focal elements and normalize, but we loose information on partial ignorances

Pignistic transform redistributes mass of partial ignorances equally to singletons included in them [Smets 1990]

$$P\{A\} = \sum_{X \in 2^\Theta} \frac{|X \cap A|}{|X|} m(X)$$

DSmP transform redistributes mass of partial ignorances proportionally to masses of singletons included in them [Dezert-Smarandache 2008]

$$\forall X \in G^\Theta \setminus \{\emptyset\} \qquad DSmP_\epsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum_{\substack{Z \subseteq X \cap Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(X \cap Y)}{\sum_{\substack{Z \subseteq Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(Y)} m(Y)$$

$\epsilon \geq 0$ is a tuning parameter

Qualitative BetP and DSmP are possible. Other transforms exist.

Example for BetP and DSmP

Example BetP versus DSmP

	θ_1	θ_2	$\theta_1 \cup \theta_2$
$m(.)$	0.3	0.1	0.6

Shafer's model

Shannon's entropy $H(P) \triangleq -\sum_{i=1}^{n} P\{\theta_i\} \log_2(P\{\theta_i\})$

With BetP

$$\begin{cases} BetP(\emptyset) = 0 \\ BetP(\theta_1 \cap \theta_2) = 0 \\ BetP(\theta_1) = m(\theta_1) + \frac{1}{2} m(\theta_1 \cup \theta_2) = 0.6 \\ BetP(\theta_2) = m(\theta_2) + \frac{1}{2} m(\theta_1 \cup \theta_2) = 0.4 \\ BetP(\theta_1 \cup \theta_2) = m(\theta_1) + m(\theta_2) + m(\theta_1 \cup \theta_2) = 1 \end{cases}$$

H(BetP)=0.9710 bits **Bigger entropy**

With DSmP

$$\begin{cases} DSmP_{\epsilon=0.001}(\emptyset) = 0 \\ DSmP_{\epsilon=0.001}(\theta_1 \cap \theta_2) = 0 \\ DSmP_{\epsilon=0.001}(\theta_1) = m(\theta_1) + \frac{m(\theta_1)+\epsilon}{m(\theta_1)+m(\theta_2)+2\epsilon} \cdot m(\theta_1 \cup \theta_2) = 0.7492 \\ DSmP_{\epsilon=0.001}(\theta_2) = m(\theta_2) + \frac{m(\theta_2)+\epsilon}{m(\theta_1)+m(\theta_2)+2\epsilon} \cdot m(\theta_1 \cup \theta_2) = 0.2508 \\ DSmP_{\epsilon=0.001}(\theta_1 \cup \theta_2) = m(\theta_1) + m(\theta_2) + m(\theta_1 \cup \theta_2) = 1 \end{cases}$$

H(DSmP)=0.8125 bits **Lower entropy**

Distances between two BBAs

[Han D., Dezert J., Yang Y., Belief interval Based Distances Measures in the Theory of Belief Functions, to appear in IEEE Trans. on SMC, 2017.]

1993 - Tessem's distance - Not a strict metric

$$d_T(m_1, m_2) = \max_{A \subseteq \Theta} \{|\mathrm{BetP}_1(A) - \mathrm{BetP}_2(A)|\}$$

2001 - Jousselme's distance - A strict metric proved in [Bouchard et al. in 2013]

$$d_J(m_1, m_2) = \sqrt{\frac{1}{2} \cdot (m_1 - m_2)^T \mathbf{Jac}\ (m_1 - m_2)}$$

$$\mathbf{Jac}(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

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New distances between 2 BBAs

2014 - Euclidean belief interval based distance [Han-Dezert-Yang 2014]

$$d_{BI}^{E}(m_1,m_2)=\sqrt{N_c\cdot\sum_{i=1}^{2^n-1}[d^I(BI_1(A_i),BI_2(A_i))]^2} \qquad N_c=1/2^{n-1}$$

2014 - Chebyshev belief interval based distance

$$d_{BI}^{C}(m_1,m_2)=\max_{A_i\subseteq\Theta,i=1,\ldots,2^n-1}\left\{d^I\left(BI_1(A_i),BI_2(A_i)\right)\right\}$$

using **Wasserstein's distance of interval numbers**

$$d^I\left([a_1,b_1],[a_2,b_2]\right)=\sqrt{\left[\frac{a_1+b_1}{2}-\frac{a_2+b_2}{2}\right]^2+\frac{1}{3}\left[\frac{b_1-a_1}{2}-\frac{b_2-a_2}{2}\right]^2}$$

because belief intervals BI=[Bel(.),Pl(.]=[a,b] are just interval numbers.

Example of distances results

Example

$$m_1(\{\theta_1\}) = m_1(\{\theta_2\}) = m_1(\{\theta_3\}) = 1/3;$$
$$m_2(\{\theta_1\}) = m_2(\{\theta_2\}) = m_2(\{\theta_3\}) = 0.1, m_2(\Theta) = 0.7;$$
$$m_3(\{\theta_1\}) = m_3(\{\theta_2\}) = 0.1, m_2(\theta_3) = 0.8.$$

Distance types	d_J	d_T	d_F	d_C	d_{BI}^E	d_{BI}^C
$d(m_1, m_2)$	0.4041	0	0.5833	0.2000	0.2858	0.2333
$d(m_1, m_3)$	0.4041	0.4667	0.6364	0.6667	0.4041	0.4667

Jousselme distance

seems not very reasonable (m2 makes no preference for choice, whereas m3 prefers the 3rd element)

Tessem's (BetP) distance

not intuitively acceptable because m1 different of m2 but dT(m1,m2)=0.

New belief interval distances

result makes more sense because $d(m_1, m_2) < d(m_1, m_3)$

Part 2

Decision-Making with Belief Functions

Decision-making with BF

Classical historical methods

They don't exploit all information

Pessimistic attitude: Max of Bel(.)

$$\theta_{i^\star} = \arg\max_i Bel(\theta_i)$$

Optimistic attitude: Max of Pl(.)

$$\theta_{i^\star} = \arg\max_i Pl(\theta_i)$$

Compromise attitude:

Use a probabilistic transformation to estimate a subjective proba measure P(.) in [Bel(.),Pl(.)]. Typically max of BetP, or max of DSmP.

$$\theta_{i^\star} = \arg\max_i P(\theta_i)$$

New method based on distance

We use the following distance

$$d_{BI}^{E}(m_1, m_2) = \sqrt{N_c \cdot \sum_{i=1}^{2^n-1} [d^I(BI_1(A_i), BI_2(A_i))]^2} \qquad N_c = 1/2^{n-1}$$

with **Wasserstein's distance of interval numbers**

$$d^I([a_1, b_1], [a_2, b_2]) = \sqrt{\left[\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}\right]^2 + \frac{1}{3}\left[\frac{b_1-a_1}{2} - \frac{b_2-a_2}{2}\right]^2}$$

Decision rule with its quality

Mass focused on X only

$$\hat{X} = \arg \min_{X \in 2^\Theta \setminus \{\emptyset\}} d_{BI}(m, m_X)$$

$$q(\hat{X}) \triangleq 1 - \frac{d_{BI}(m, m_{\hat{X}})}{\sum_{X \in 2^\Theta \setminus \{\emptyset\}} d_{BI}(m, m_X)}$$

higher is q, better is the decision. q=1 is max if numerator is zero

[J. Dezert, D. Han, J.-M. Tacnet, S. Carladous, Y. Yang, Decision-Making with Belief Interval Distance, in Proc. of Belief 2016.]

General decision-making problem

Mono-criterion case

States of the nature

Alternatives

$$\begin{array}{cc} & \begin{array}{ccccc} S_1 & \cdots & S_j & \cdots & S_n \end{array} \\ \begin{array}{c} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} & \begin{pmatrix} C_{11} & \cdots & C_{11} & \cdots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \cdots & C_{ij} & \cdots & C_{in} \\ & & \vdots & & \\ C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} \end{pmatrix} = C \end{array}$$

benefit/payoff matrix

How to select the best alternative A^* given C matrix and the knowledge one has on the states of the nature?

Decision frameworks

Decision under certainty

If we know the true state of nature is S_j take

$$A^* = A_{i^*} \quad \text{with} \quad i^* \triangleq \arg\max_i \{C_{ij}\}$$

Decision under risk

If we know all probas $p_j=P(S_j)$, then compute expected benefits $E[C_i] = \sum_j p_j \cdot C_{ij}$ and take

$$A^* = A_{i^*} \quad \text{with} \quad i^* \triangleq \arg\max_i \{E[C_i]\}$$

Decision under ignorance

If we don't know probabilities $p_j=P(S_j)$, use Yager's OWA (Ordered Weighted Averaging) approach (1988).

Decision under uncertainty

If we have only a BBA defined on the power-set 2^S, where $S=\{S_1, S_2,..., S_n\}$, Yager proposed extended OWA.

[R. Yager, On ordered weighted averaging operators in multi-criteria decision making, IEEE Trans. on SMC, 18:183–190, 1988.]

Decision under ignorance

Yager's OWA method

p_j=??? => P(S_j) are **unknown**

Alternatives

$$\begin{array}{c|c} & \begin{array}{ccccc} S_1 & \cdots & S_j & \cdots & S_n \end{array} \\ \begin{array}{c} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} & \begin{pmatrix} C_{11} & \cdots & C_{11} & \cdots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \cdots & C_{ij} & \cdots & C_{in} \\ & & \vdots & & \\ C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} \end{pmatrix} = C \end{array}$$

Step 1 (Decisional attitude) Choose a normalized set of weights $w_{i1}, \ldots, w_{in}$ with $w_{i1} + \ldots + w_{in} = 1$

Step 2 (Evaluation) Compute the weighted average of ordered benefits for each row (alternative) i=1,2, ...q

$$V_i \triangleq \text{OWA}(C_{i1}, C_{i2}, \ldots, C_{in}) = \sum_j w_{ij} \cdot b_{ij}$$

b_{ij} is the jth largest element in the collection of benefit {C_{i1}, ... C_{in}}

[bi1,....bin] = reorder of ith row by decreasing values

Step 3 (Decision) Select $A^* = A_{i^*}$ with $i^* \triangleq \arg\max_i \{V_i\}$

Decision under ignorance (example)

$p_j = P(S_j)$ are unknown

$$C = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{pmatrix} 10 & 0 & 20 & 30 \\ 1 & 10 & 20 & 30 \\ 30 & 10 & 2 & 5 \end{pmatrix} \end{array}$$

1 Pessimistic choice (we take the min value per row)

$$W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$V_1 = \mathrm{OWA}(10, 0, 20, 30) = 0$$
$$V_2 = \mathrm{OWA}(1, 10, 20, 30) = 1$$
$$V_3 = \mathrm{OWA}(30, 10, 2, 5) = 2$$

Best choice = A_3

2 Optimitic choice

$$W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_1 = \mathrm{OWA}(10, 0, 20, 30) = 30$$
$$V_2 = \mathrm{OWA}(1, 10, 20, 30) = 30$$
$$V_3 = \mathrm{OWA}(30, 10, 2, 5) = 30$$

All alternatives have same score

3 Hurwicz choice $\alpha = 0.3$ (balance between min and max values per row)

$$W = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 1-\alpha \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0 \\ 0 \\ 0.7 \end{bmatrix}$$

$$V_1 = \mathrm{OWA}(10, 0, 20, 30) = 9$$
$$V_2 = \mathrm{OWA}(1, 10, 20, 30) = 9.7$$
$$V_3 = \mathrm{OWA}(30, 10, 2, 5) = 10.4$$

Best choice = A_3

4 Normative choice

$$W = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$V_1 = \mathrm{OWA}(10, 0, 20, 30) = 60/4 = 15$$
$$V_2 = \mathrm{OWA}(1, 10, 20, 30) = 61/4$$
$$V_3 = \mathrm{OWA}(30, 10, 2, 5) = 47/4$$

Best choice = A_2

Decision under uncertainty

Yager's OWA method

We have only a BBA m(.) over 2^S of states $S=\{S_1, S_2,\ldots, S_n\}$

Alternatives

$$
\begin{array}{c|ccccc|l}
 & S_1 & \cdots & S_j & \cdots & S_n & \\
\hline
A_1 & C_{11} & \cdots & C_{11} & \cdots & C_{1n} & \\
\vdots & & & \vdots & & & \\
A_i & C_{i1} & \cdots & C_{ij} & \cdots & C_{in} & = C \\
\vdots & & & \vdots & & & \\
A_q & C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} &
\end{array}
$$

Step 1 (Decisional attitude) Choose weights $w_{i1}, \ldots, w_{in}$ with $w_{i1} + \ldots + w_{in} = 1$

Step 2 (Evaluation) For each benefit subrow M_{ik} associated to a focal element X_k of BBA m(.) compute the benefit of V_{ik} of A_i by

$$V_{ik} = \mathrm{OWA}(M_{ik}) \quad \text{and} \quad M_{ik} = \{C_{ij} | S_j \in X_k\}$$

Compute generalized expected benefits $$E[C_i] = \sum_{k=1}^{r} m(X_k) V_{ik}$$

Step 3 (Decision) Select $$A^* = A_{i^*} \quad \text{with} \quad i^* = \arg\max_i E[C_i]$$

Decision under uncertainty (example)

States of the world
$S = \{S_1, S_2, S_3, S_4, S_5\}$

Alternatives
$A = \{A_1, A_2, A_3, A_4\}$

$$\underbrace{m(S_1 \cup S_3 \cup S_4)}_{X_1} = 0.6 \qquad \underbrace{m(S_2 \cup S_5)}_{X_2} = 0.3 \qquad \underbrace{m(S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5)}_{X_3} = 0.1$$

$X_1 \leftarrow$ partial ignorances $\rightarrow X_2$ full ignorance $\rightarrow X_3$

$$C = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{pmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{pmatrix} \end{array}$$

$$M(X_1) = \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \\ M_{41} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} S_1 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \end{array}$$

$$M(X_2) = \begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \\ M_{42} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} S_1 & S_5 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \end{array}$$

$$M(X_3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \\ M_{43} \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} = C$$

sub-payoff matrices

Decision under uncertainty (ex. cont'd)

Pessimist decision

Pessimistic attitude One takes the min value by row

$$M(X_1) = \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \\ M_{41} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} S_1 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \end{array} \quad W = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad V(X_1) = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 6 \end{bmatrix}$$

$$X_1 = S_1 \cup S_3 \cup S_4 \qquad m(X_1) = 0.6$$

$$M(X_2) = \begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \\ M_{42} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} S_1 & S_5 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \end{array} \quad W = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad V(X_2) = \begin{bmatrix} 5 \\ 2 \\ 9 \\ 4 \end{bmatrix}$$

$$X_2 = S_2 \cup S_5 \qquad m(X_2) = 0.3$$

$$M(X_3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \\ M_{43} \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} = C \quad W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad V(X_3) = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X_3 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \qquad m(X_3) = 0.1$$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{bmatrix} E[C_1] \\ E[C_2] \\ E[C_3] \\ E[C_4] \end{bmatrix} = \sum_{k=1}^{r=3} m(X_k) \cdot V(X_k) = \begin{bmatrix} 6.2 \\ 3.8 \\ 4.8 \\ 5.2 \end{bmatrix}$$

$$7 \cdot 0.6 + 5 \cdot 0.3 + 5 \cdot 0.1 = 6.2 \Leftarrow \boxed{A_1}$$

Best choice = A_1

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Decision under uncertainty (ex. cont'd)

Optimist decision

Optimistic attitude

One takes the max value by row

$$M(X_1) = \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \\ M_{41} \end{bmatrix} = \begin{array}{ccc} S_1 & S_3 & S_4 \\ \end{array} \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \quad W = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad V(X_1) = \begin{bmatrix} 13 \\ 12 \\ 10 \\ 15 \end{bmatrix} \quad X_1 = S_1 \cup S_3 \cup S_4 \quad m(X_1) = 0.6$$

$$M(X_2) = \begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \\ M_{42} \end{bmatrix} = \begin{array}{cc} S_1 & S_5 \end{array} \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \quad W = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad V(X_2) = \begin{bmatrix} 6 \\ 10 \\ 13 \\ 9 \end{bmatrix} \quad X_2 = S_2 \cup S_5 \quad m(X_2) = 0.3$$

$$M(X_3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \\ M_{43} \end{bmatrix} = \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} = C \quad W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad V(X_3) = \begin{bmatrix} 13 \\ 12 \\ 13 \\ 15 \end{bmatrix} \quad X_3 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \quad m(X_3) = 0.1$$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{bmatrix} E[C_1] \\ E[C_2] \\ E[C_3] \\ E[C_4] \end{bmatrix} = \sum_{k=1}^{r=3} m(X_k) \cdot V(X_k) = \begin{bmatrix} 10.9 \\ 11.4 \\ 11.2 \\ 13.2 \end{bmatrix}$$

$$13 \cdot 0.6 + 6 \cdot 0.3 + 13 \cdot 0.1 = 10.9$$

$\boxed{A_4}$ Best choice = A_4

(A_4 is also chosen with normative attitude)

Improvement of OWA method

Drawbacks of Yager's OWA approach

Result strongly depends on the decisional attitude. How to avoid this?

Solution

Use jointly the two most extreme attitudes (pessimistic and optimistic) to be more cautious.

Cautious OWA (COWA) method [Tacnet-Dezert 2011]

Pessimistic and optimistic generalized expected benefits allow to build belief intervals, and to get BBAs that are combined with PCR6 to get combined BBA to take final decision..

[Tacnet J.-M., Dezert J., Cautious OWA and Evidential Reasoning for Decision Making under Uncertainty, in Proc. Of Fusion 2011.]

Fuzzy COWA method (simpler than COWA)

[Han D., Dezert J., Tacnet J.-M., Han C., A Fuzzy-Cautious OWA Approach with Evidential Reasoning, in Proc. Of Fusion 2012, Singapore, July 2012.]

Part 3

Multi-Criteria Decision-Making Support

- DSm-AHP
- BF-TOPSIS

Classical AHP method

Analytic Hierarchy Process (AHP) is a Multi-criteria decision-making method developed by **Thomas Saaty** in 1980’s based on the derivation of priority from preferences.

[T.L. Saaty,The Analytical Hierarchy Process,McGraw Hill, 1980.]

Principle of AHP

1) The multiple criteria are ordered in a hierarchy of importance. For each criterion, a set of preferences of the choice is established from pairwise comparison matrices.

2) Combine these preferences to get the global ranking of the solutions (in classical AHP, this is done by the **weighted arithmetic mean**).

3) Final decision-making based on the result of step 2.

Classical AHP method (example)

Example (Four cars and 3 criteria): $\{Cars\} = \Theta = \{A, B, C, D\}$

$\{\text{Criteria}\} = \{C_1 = \text{Fuel economy}, C_2 = \text{Reliability}, C_3 = \text{Style}\}$

Priorities = Normalized Perron-Frobenius vector

Ranking criteria: $\mathbf{M} = \begin{bmatrix} 1/1 & 1/3 & 4/1 \\ \boxed{3/1} & 1/1 & \boxed{5/1} \\ 1/4 & 1/5 & 1/1 \end{bmatrix} \xrightarrow[\text{associated to largest eigenvalue}]{\text{normalized eigenvector}} \mathbf{w} = \begin{bmatrix} 0.2797 \\ 0.6267 \\ 0.0936 \end{bmatrix}$

C2 is the most important criterion

$M_{21}=3/1$ means «Reliability criterion is 3 times as important as Fuel Economy criterion».

$M_{23}=5/1$ means «Reliability criterion is 5 times as important as Style criterion».

Ranking cars w.r.t. C_i

A similar procedure is applied to rank cars for each criterion

$$[\mathbf{w}(C_1)\ \mathbf{w}(C_2)\ \mathbf{w}(C_3)] = \begin{array}{c|c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \\ \begin{bmatrix} 0.2500 & 0.4733 & 0.1129 \\ 0.1304 & 0.0611 & 0.4435 \\ 0.5109 & 0.1832 & 0.0565 \\ 0.1087 & 0.2824 & 0.3871 \end{bmatrix} & \begin{array}{c} \text{car } A \\ \text{car } B \\ \text{car } C \\ \text{car } D \end{array} \end{array}$$

Final AHP result

$$\begin{bmatrix} 0.2500 & 0.4733 & 0.1129 \\ 0.1304 & 0.0611 & 0.4435 \\ 0.5109 & 0.1832 & 0.0565 \\ 0.1087 & 0.2824 & 0.3871 \end{bmatrix} \times \begin{bmatrix} 0.2797 \\ 0.6267 \\ 0.0936 \end{bmatrix} = \begin{bmatrix} \boxed{0.3771} \\ 0.1163 \\ 0.2630 \\ 0.2436 \end{bmatrix}$$

car A is the best choice

DSm-AHP method

Extension of Analytic Hierarchy Process (AHP) with BF, PCR rules and importance discounting technique to take into account uncertainty (previous attempts by Beynon in 2000).

[Dezert J, Tacnet J.-M., Batton-Hubert M., Smarandache F., Multi-criteria decision making based on DSmT/AHP, Proc. of Belief 2010.]

[Dezert J., Tacnet J.-M., Evidential Reasoning for Multi-Criteria Analysis based on DSmT-AHP, Proc. of ISAHP 2011.]

Principle of DSm-AHP

1) Construction of uncertain comparison matrices.
   Take as BBA, the normalized Perron-Frobenius vector of each matrix
2) Use PCR6, to combine BBA's to get a final priority ranking.
3) Decision-making from combined result (max of Bel, Pl, BetP(.), DSmP, etc, or by min of distance).

Need for importance discounting

Importance discounting

Reliability discounting

$$\begin{cases} m_\alpha(X) = \alpha \cdot m(X), & \text{for } X \neq \Theta \\ m_\alpha(\Theta) = \alpha \cdot m(\Theta) + (1-\alpha) \end{cases}$$

$\neq$

Importance discounting

$$\begin{cases} m_\beta(X) = \beta \cdot m(X), & \text{for } X \neq \emptyset \\ m_\beta(\emptyset) = \beta \cdot m(\emptyset) + (1-\beta) \end{cases}$$

ß factors = Importances of criteria

PCR5/6 fusion rules

Use importance discounting technique combined with non normalized version of PCR5/6

$$m_{PCR5_\emptyset}(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap X = \emptyset}} [\frac{m_1(X)^2 m_2(X_2)}{m_1(X)+m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X)+m_1(X_2)}]$$

then a classical normalization applies.

Note: Demspter's rule doesn't react to importance discounting!

[Smarandache F., Dezert J., Tacnet J.-M., Fusion of sources of evidence with different importances and reliabilities, Proc. Fusion 2010.]

DSm-AHP example

Example (Three cars and 2 criteria): $\{Cars\} = \Theta = \{A, B, C\}$

$$\{\text{Criteria}\} = \{C_1 = \text{Fuel economy}, C_2 = \text{Reliability}\}$$

Let us assume the following (uncertain) comparisons w.r.t each criterion

$$\mathbf{M}(C1) = \left[\begin{array}{c|ccc} & A & B\cup C & \Theta \\ \hline A & 1 & 0 & 1/3 \\ B\cup C & 0 & 1 & 2 \\ \Theta & 3 & 1/2 & 1 \end{array}\right] \longrightarrow \mathbf{w}(C1) \approx \begin{bmatrix} 0.0889 \\ 0.5337 \\ 0.3774 \end{bmatrix}$$

$$\mathbf{M}(C2) = \left[\begin{array}{c|cccc} & A & B & A\cup C & B\cup C \\ \hline A & 1 & 2 & 4 & 3 \\ B & 1/2 & 1 & 1/2 & 1/5 \\ A\cup C & 1/4 & 2 & 1 & 0 \\ B\cup C & 1/3 & 5 & 0 & 1 \end{array}\right] \longrightarrow \mathbf{w}(C2) \approx \begin{bmatrix} 0.5002 \\ 0.1208 \\ 0.1222 \\ 0.2568 \end{bmatrix}$$

Elem. of 2^Θ	$m_{C1}(.)$	$m_{C2}(.)$
$\emptyset$	0	0
A	0.0889	0.5002
B	0	0
$A\cup B$	0	0.1208
C	0	0
$A\cup C$	0	0.1222
$B\cup C$	0.5337	0.2568
$A\cup B\cup C$	0.3774	0

We need to fuse these bba's taking into account the importances of criterion C_1 and C_2

DSm-AHP example (cont'd)

Case 1: If C_1 and C_2 have **same** full importance ($\beta_1=\beta_2=1$), we fuse with PCR5/PCR6

Elem. of 2^Θ	$m_{C1}(.)$	$m_{C2}(.)$	$m_{PCR5}(.)$
$\emptyset$	0	0	0
A	0.0889	0.5002	0.3837
B	0	0	0.1162
$A \cup B$	0	0.1208	0
C	0	0	0.0652
$A \cup C$	0	0.1222	0.0461
$B \cup C$	0.5337	0.2568	0.3887
$A \cup B \cup C$	0.3774	0	0

Final decision based on max of

Elem. of Θ	$Bel(.)$	$BetP(.)$	$Pl(.)$
A	0.3837	0.4068	0.4298
B	0.1162	0.3105	0.5049
C	0.0652	0.2826	0.5000

Case 2: C_1 and C_2 have **different** importances

$$\{\text{Criteria}\} = \{C_1 = \text{Fuel economy}, C_2 = \text{Reliability}\}$$

user preferences

importances

$$\text{Ranking criteria:} \quad \mathbf{M} = \begin{bmatrix} 1/1 & 1/3 \\ 3/1 & 1/1 \end{bmatrix} \longrightarrow \mathbf{w} = \begin{bmatrix} 0.2500 \to \beta_1 \\ 0.7500 \to \beta_2 \end{bmatrix}$$

We apply importance discounting to get bba's

DSm-AHP example (cont'd)

Case 2: C_1 and C_2 have **different** importances (cont'd)

Result with Dempster's rule

importance has no impact !!!!

Elem. of 2^Θ	$m_{DS}(.)$	$m_{DS,\mathbf{w}}(.)$
$\emptyset$	0	0
A	0.3588	0.3588
B	0.0908	0.0908
$A \cup B$	0.0642	0.0642
C	0.0918	0.0918
$A \cup C$	0.0649	0.0650
$B \cup C$	0.3294	0.3294
$A \cup B \cup C$	0	0

Result with importance and PCR5/PCR6

Elem. of 2^Θ	$m_{PCR5_\emptyset}(.)$	$m^{normalized}_{PCR5_\emptyset}(.)$
$\emptyset$	0.6558	0
A	0.1794	0.5213
B	0.0121	0.0351
$A \cup B$	0.0159	0.0461
C	0.0122	0.0355
$A \cup C$	0.0161	0.0469
$B \cup C$	0.1020	0.2963
$A \cup B \cup C$	0.0065	0.0188

Final decision based on max of

Elem. of Θ	$Bel(.)$	$BetP(.)$	$Pl(.)$
A	0.5213	0.5741	0.6331
B	0.0351	0.2126	0.3963
C	0.0355	0.2134	0.3974

Result with direct AHP using bba's

$$m_{AHP}(.) = \begin{bmatrix} 0 & 0 \\ 0.0889 & 0.5002 \\ 0 & 0 \\ 0 & 0.1208 \\ 0 & 0 \\ 0 & 0.1222 \\ 0.5337 & 0.2568 \\ 0.3774 & 0 \end{bmatrix} \times \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3974 \\ 0 \\ 0.0906 \\ 0 \\ 0.0917 \\ 0.3260 \\ 0.0944 \end{bmatrix}$$

Final decision based on max of

Elem. of Θ	$Bel(.)$	$BetP(.)$	$Pl(.)$
A	0.3974	0.5200	0.6741
B	0	0.2398	0.5110
C	0	0.2403	0.5121

DSmT/AHP reduces uncertainty

$$U(X) = Pl(X) - Bel(X)$$

Elem. of Θ	$U(.)$ with AHP	$U(.)$ with DSmT-AHP
A	0.2767	0.1118
B	0.5110	0.3612
C	0.5121	0.3619

BF-TOPSIS Methods

[J. Dezert, D. Han, H. Yin, A new belief function based approach for multi-criteria decision-making support, in Proc. of Fusion 2016.]

[J. Dezert, D. Han, J.-M. Tacnet, Multi-Criteria Decision-Making with Imprecise Scores and BF-TOPSIS, in Proc. of Fusion 2017.]

How to choose an alternative (make a choice) among a known set of alternatives based on their numerical scores ?

What we have

Alternatives: $\mathbf{A} \triangleq \{A_1, A_2, \ldots, A_M\}$ $(M > 2)$

Criteria: $\mathbf{C} \triangleq \{C_1, C_2, \ldots, C_N\}$ $(N \geq 1)$

Score matrix: $\mathbf{S} \triangleq [S_{ij}]$

Difficulties

Scores are given in different units and different scales (how to normalize them?)

Criteria C_j do not have same weights w_j of importance in general. $w_j \in [0, 1]$, $j = 1, \ldots, N$ and $\sum_j w_j = 1$.

BF-TOPSIS Methods

Some facts

- Most of existing methods require score normalization at first, but ERV (Estimator Ranking Vector) method [Yin et al. Fusion 2013].
- They all suffer of rank reversal problem (the rank is changed by adding or deleting an alternative) which is a serious drawback.
- None of them makes consensus among users.
- TOPSIS (technique for order preference by similarity to ideal solution) method is widely used but it requires data normalization.

Our goals

- Develop a new TOPSIS method without score normalization to be more robust to rank reversal based on belief functions (BF).

Our contribution

- A new generic approach for MCDM, called BF-TOPSIS.
- A new method for building BF from unnormalized scores.
- Four BF-TOPSIS methods are proposed (having different complexities).

BF-TOPSIS Methods

How to construct BBAs on FoD of alternatives $\mathbf{A} \triangleq \{A_1, A_2, \ldots, A_M\}$ from the given score matrix $\mathbf{S} \triangleq [S_{ij}]$?

We define the positive support of A_i based on all scores values of criteria C_j to measure how much A_i is better than other alternatives as follows

$$Sup_j(A_i) \triangleq \sum_{k \in \{1,\ldots M\} | S_{kj} \leq S_{ij}} |S_{ij} - S_{kj}|$$

We define the negative support of A_i based on all scores values of criteria C_j to measure how much A_i is worse than other alternatives as follows

$$Inf_j(A_i) \triangleq - \sum_{k \in \{1,\ldots M\} | S_{kj} \geq S_{ij}} |S_{ij} - S_{kj}|$$

We prove that the inequality always holds (see Theorem and its proof in the paper)

$$\frac{Sup_j(A_i)}{A^j_{\max}} \leq 1 - \frac{Inf_j(A_i)}{A^j_{\min}}$$

iff $A^j_{\max} \triangleq \max_i Sup_j(A_i)$ and $A^j_{\min} \triangleq \min_i Inf_j(A_i)$ are different from zero.

BF-TOPSIS Methods

Belief modeling

The belief of A_i and of $\bar{A}_i$ are defined by

$$Bel_{ij}(A_i) \triangleq \frac{Sup_j(A_i)}{A^j_{\max}} \quad \text{and} \quad Bel_{ij}(\bar{A}_i) \triangleq \frac{Inf_j(A_i)}{A^j_{\min}}$$

If $A^j_{\max} = 0$ (there is no evidential support for A_i), then $Bel_{ij}(A_i) = 0$.
If $A^j_{\min} = 0$ (there is no evidential support for $\bar{A}_i$), then $Bel_{ij}(\bar{A}_i) = 0$.

By construction, $Bel_{ij}(A_i)$ and $Bel_{ij}(\bar{A}_i)$ belong to $[0, 1]$, and thanks to previous inequality one always has $Bel_{ij}(A_i) \leq (Pl_{ij}(A_i) = 1 - Bel_{ij}(\bar{A}_i))$.

BBA construction

From belief intervals $[Bel_{ij}(A_i), Pl_{ij}(A_i)]$, one get $M \times N$ BBAs $m_{ij}(\cdot)$ by taking

$$m_{ij}(A_i) = Bel_{ij}(A_i)$$
$$m_{ij}(\bar{A}_i) = Bel_{ij}(\bar{A}_i) = 1 - Pl_{ij}(A_i)$$
$$m_{ij}(A_i \cup \bar{A}_i) = Pl_{ij}(A_i) - Bel_{ij}(A_i)$$

BF-TOPSIS Methods

How to use BBAs matrix M $= [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]$ **to rank alternatives?**

- Direct fusion of BBAs by DS or PCR6 rules does not work efficiently (see example 1)
- We develop a modified TOPSIS approach with BF $\Rightarrow$ 4 BF-TOPSIS methods.

BF-TOPSIS1 method
For each A_i, we compute the weighted average distances to ideal best and worst solutions, and from them the closeness measures $C(A_i, A^{\text{best}})$ to A^{best}.

BF-TOPSIS2 method
For each criteria C_j, we compute the closeness measures $C_j(A_i, A^{\text{best}})$ to ideal best A^{best} solution, and then we make the weighted average of closeness measures to get $C(A_i, A^{\text{best}})$.

BF-TOPSIS3 method
For each A_i, we fuse BBAs of the same row of matrix **M** with PCR6, then compute distances to ideal solutions to get relative closeness measure to ideal best solution.

BF-TOPSIS4 method
Same as BF-TOPSIS3 using ZPCR6 (PCR6 with Zhang's degree of intersection).

BF-TOPSIS Methods

BF-TOPSIS1 method

Step 1: From **S**, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$
Step 2: Set $m_{ij}^{\text{best}}(A_i) \triangleq 1$, and $m_{ij}^{\text{worst}}(\bar{A}_i) \triangleq 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})$ to ideal solutions.
Step 3: Compute the weighted average distances for A_i, $i = 1, \ldots, M$

$$d^{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})$$

$$d^{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})$$

Step 4: Compute the relative closeness of A_i, $i = 1, \ldots, M$, with respect to ideal best solution A^{best}

$$C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)}$$

Step 5: Sort $C(A_i, A^{\text{best}}) \in [0, 1]$ in descending order (larger $C(A_i, A^{\text{best}})$ means higher preference).

BF-TOPSIS Methods

Example 1 (Mono-criteria): Preference order $\rightarrow$ greater value is better

$$\mathbf{S} \triangleq \begin{array}{c|c} & C_1 \\ \hline A_1 & 10 \\ A_2 & 20 \\ A_3 & -5 \\ A_4 & 0 \\ A_5 & 100 \\ A_6 & -11 \\ A_7 & 0 \end{array} \Rightarrow \mathbf{M} \triangleq \begin{array}{c|ccc} & m_{i1}(A_i) & m_{i1}(\bar{A}_i) & m_{i1}(A_i \cup \bar{A}_i) \\ \hline A_1 & 0.0955 & 0.5236 & 0.3809 \\ A_2 & 0.1809 & 0.4188 & 0.4003 \\ A_3 & 0.0102 & 0.8115 & 0.1783 \\ A_4 & 0.0273 & 0.6806 & 0.2921 \\ A_5 & 1.0000 & 0 & 0 \\ A_6 & 0 & 1.0000 & 0 \\ A_7 & 0.0273 & 0.6806 & 0.2921 \end{array} \Rightarrow \begin{array}{c|c} & C(A_i, A^{\text{best}}) \\ \hline A_1 & 0.1130 \\ A_2 & 0.1948 \\ A_3 & 0.0257 \\ A_4 & 0.0485 \\ A_5 & 1.0000 \\ A_6 & 0 \\ A_7 & 0.0485 \end{array}$$

Ranking methods	Preferences order
By direct ranking	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By BF-TOPSIS	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By DS fusion	$A_5 \succ (A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_6 \sim A_7)$
By PCR6 fusion	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ (A_3 \sim A_6 \sim A_7)$

Rankings based on DS and PCR6 fusion do not match with direct ranking even in mono criteria case because of dependencies of BBAs in their construction.

BF-TOPSIS Methods

Example 2 (Mono-criteria): Non informative case

When all scores are the same:

$\Rightarrow$ all BBAs are the same and equal to the vacuous BBA

$\Rightarrow$ all closeness measures to best ideal solution are equal

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} C_1 \\ \begin{bmatrix} s \\ \vdots \\ s \\ \vdots \\ s \end{bmatrix} \end{array} \Rightarrow \mathbf{M} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} m_{i1}(A_i \cup \bar{A}_i) \\ \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{array} \Rightarrow \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} C(A_i, A^{\text{best}}) \\ \begin{bmatrix} c \\ \vdots \\ c \\ \vdots \\ c \end{bmatrix} \end{array}$$

Conclusion: No specific choice can be drawn, which is perfectly normal.

BF-TOPSIS Methods

Example 3 (Multi-criteria): 5 alternatives, 4 criteria [Wang-Luo-2009]

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1, \frac{1}{6} & C_2, \frac{1}{3} & C_3, \frac{1}{3} & C_4, \frac{1}{6} \end{array} \\ \begin{bmatrix} 36 & 42 & 43 & 70 \\ 25 & 50 & 45 & 80 \\ 28 & 45 & 50 & 75 \\ 24 & 40 & 47 & 100 \\ 30 & 30 & 45 & 80 \end{bmatrix} \end{array}$$

Set of alternatives	TOPSIS
$\{A_1, A_2, A_3\}$ $\{A_1, A_2, A_3, A_4\}$	$A_3 \succ A_2 \succ A_1$ $A_2 \succ A_3 \succ A_1 \succ A_4$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal

Set of alternatives	BF-TOPSIS1 & 2	BF-TOPSIS3 & 4
$\{A_1, A_2, A_3\}$ $\{A_1, A_2, A_3, A_4\}$	$A_2 \succ A_3 \succ A_1$ $A_3 \succ A_2 \succ A_4 \succ A_1$	$A_3 \succ A_2 \succ A_1$ $A_3 \succ A_2 \succ A_4 \succ A_1$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal	No rank reversal

Thank you for your attention.

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THE FRENCH AEROSPACE LAB

return on innovation

Short biography

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Jean Dezert was born in l'Hay les Roses, France, on August 25, 1962. He received the Electrical Engineering (EE) degree in 1985, and his Ph.D. from the University Paris XI, Orsay, in 1990 in Automatic Control and Signal Processing. During 1986-1990 he was with the Syst. Dept. at the French Aerospace Lab (ONERA) and did research in multi-sensor multi-target tracking (MS-MTT). During 1991-1992, he visited the Dept. of ESE, UConn, Storrs, USA as an ESA Postdoctoral Research Fellow under supervision of Prof. Bar-Shalom. During 1992-1993 he was teaching assistant in EE Dept, University of Orléans, France. Since 1993, he is Senior Research Scientist in the Information Processing and Modeling Department at the French Aerospace Lab. His current research interests include estimation theory, and information fusion (IF) and plausible reasoning and multi-criteria decision-making support with applications to MS-MTT, defense and security, robotics and risk assessment. Jean Dezert has been involved within International Society of Information Fusion (ISIF – www.isif.org) since its beginning and has been the Local Arrangements Co-Organizer of the first Fusion Conference in Europe in 2000. He is currently member of Executive board of ISIF (Vice-president 2004, President 2016). He has been involved in the Technical Program Committees of Fusion 2001-2017 Conf., and in several sessions and panel discussions on reasoning under uncertainty and data fusion. Jean Dezert is the co-founder with Prof. Smarandache of DSmT (Dezert-Smarandache Theory) of information fusion based on belief functions. Jean Dezert has published more than 150 papers in conferences and journals on tracking and information fusion and he has co-edited four books (collected works) in english (the first volume has been translated in chinese) devoted to DSmT. More than twenty five theses related with DSmT and its applications have been defended so far in Europe, China, USA and Canada. Jean Dezert has given tutorials, seminars and workshops in the information fusion and target tracking fields in North America, Europe, Australia and China. Jean Dezert is reviewer for several international journals and Associate Editor of ISIF Journal of Advances in Information Fusion.

References available at http://www.onera.fr/staff/jean-dezert?page=2